Joint Centre for Mesoscale Meteorology, Reading, UK

Waves in the Atmosphere and Ocean

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Summary notes

for a lecture course on

WAVES IN THE ATMOSPHERE AND OCEAN

by

A A White

JCMM/Meteorological Office

CONTENTS

Preface	1
Narrative overview of the lectures	2
Summary and list of recommended texts	6
L1 Review of basic concepts and terminology	8
L2 The classical wave equation	10
L3 Dispersion relations; neutral, amplifying and decaying waves; group velocity	12
L4 The equations of fluid motion; perfect gas and incompressible fluid	14
L5 Further aspects of the equations of fluid motion; linearisation	16
L6 Surface waves on water: dispersion relation; deep and shallow limits	18
L7 Pure acoustic waves; shallow water isomorphism; general solution	20
L8 Acoustic waves in the presence of gravity; Lamb waves	22
L9 Internal gravity waves in a stratified incompressible fluid	24
L10 Stratified incompressible flow over wavy terrain	26
L11 More on flow over wavy terrain; buoyancy oscillations in a stratified perfect gas atmosphere	28
L12 Waves in an isothermal, perfect gas atmosphere: the perturbation equations	30
L13 Waves in an isothermal, perfect gas atmosphere: dispersion relation for mixed acoustic/gravity waves; soundproofing	32
L13A Applications and implications of the acoustic/gravity wave dispersion equation; the hydrostatic and anelastic approximations	34
L14 The effects of frame rotation: Poincaré, Kelvin and inertio-gravity waves	36
L15 Growing waves: the Kelvin-Helmholtz and Rayleigh-Taylor problems	38
L15 Annex (i) Conditions at the interface between two incompressible liquids (ii) Analysis for the Rayleigh-Taylor instability problem	40
L16 Waves in inhomogeneous basic states	42
L16 Diagrams	44
L17 Rossby waves, non-normal modes and barotropic instability	48
L17 Supplementary notes: pseudomomentum and pseudoenergy; a barotropic instability criterion; adjointness	50
L20 Large amplitude effects	52

The lecture numbers (**Ln**) *mostly refer to the narrative overview* (pp 2-5) *and not to the actual lecture periods.* **L13A** *was introduced to emphasise the application and importance of the results of* **L13**. *The material of lectures 16 –20 in the narrative overview was compressed into lectures* **L16**, **L17**, **L20**.

Preface

"Waves in the Atmosphere and Ocean" was delivered at Reading University during Autumn 1998. It was an examined course of 20 lectures, offered as an option in Part III BSc Meteorology, and attended by some 15 third-year undergraduate students.

This JCMM report contains the summary notes which accompanied the course. Some minor errors in the original notes have been corrected, but the notes are substantially as presented to the class lecture by lecture. A few non-uniformities of notation have been allowed to remain, since to have ironed them out would have taken more time than has been available given my recent return to Met Office headquarters at Bracknell; I am happy to leave the task to whomever takes over the delivery of the course!

A few paragraphs of background, explanation and comment on the notes may be of interest to the reader. An important feature is that they are *summary* notes, not comprehensive notes. An arbitrary limit of 2 sides of printed notes per lecture was set (by me) at the beginning of the course – though, it should be added, without specifying the type size to be used.

The 2-side limit was observed in the sense that no main unit of notes longer than 2 sides was produced. However, a few annexes and supplementary notes did creep in, and they are included here. Also included, without revision, is a 4-page "narrative overview" which was presented at the beginning of the course, and a briefer summary (with list of recommended texts).

In the event, the basic, examinable material of the course took 17 lecture periods to deliver rather than the planned 15. Consequently, the planned five lectures on unexamined material were reduced to three, and this material was covered in less depth than had been intended at the outset. The unexamined material is covered in the notes under lectures 16, 17, and 20 (L16, L17 and L20); this numbering preserves at least ordinal consistency with the narrative overview.

The notes contain many exercises for the student to tackle. Worked solutions were offered during the course for almost all of these, but they were handwritten and are not included here.

I would like to thank the students who attended the course. Its first delivery inevitably contained a few stressful times; the class were tolerant of these occasions and offered some useful comments both during and after the course. Also, and in particular, I would like to thank Dr Mike Pedder. Mike suggested that I give the course, guided it from being a proposal to being part of the University timetable, attended most of the lectures and asked many relevant questions. He also provided discreet encouragement at diplomatic intervals. I acknowledge with gratitude my debt to Dr John Green and Dr Bob Riddaway, whose sets of lecture notes on atmospheric waves have formed the basis of my understanding of the subject as it has developed over the years.

Andy White
March 1999

WAVES IN THE ATMOSPHERE AND OCEAN (MT728)

Narrative overview of the lectures

A brief verbal account of the course may be helpful to the student. It will certainly be helpful to the lecturer; at the time of writing, lecture 1 (L1) is 11 days away, and there isn't much down on paper or acetate yet. As well as this narrative, I'm going to try to supply mathematical notes (at least for L1 - L15) which will give the essential analysis for each lecture. I have never felt that a lecture should be an exercise in copying algebra whilst ignoring what is being said. Using this summary, the mathematical notes, and perhaps the recommended textbooks, you have to build your own view of the subject. Actually, if you really understand and appreciate this verbal summary then you have learned a great deal; but you will probably not be in a position to do much with your knowledge (and you probably won't do very well in the exams).

First, some motivation. Why are we interested in waves in the atmosphere and oceans? There are at least three reasons.

i) Wave motion is the main way in which local influence due to some form of forcing (such as heating or the effect of a boundary) may spread to other parts of the atmosphere or ocean. It is desirable to understand the possibilities and mechanisms.

ii) A knowledge of the types of possible wave motions and their properties is important in the construction and operation of numerical models of fluid motion. In particular, the numerical modeller may be concerned with approximations of the governing equations which lead to the complete removal (or "filtering") of certain types of wave motion and which alter the properties of those that remain.

iii) The problems that arise are good examples of the interplay of physical and mathematical methods in classical physics.

Foundations (L1 - L5)
Five lectures may seem a lot to devote to elementary things, but we have to (i) revise basic ideas and terminology about waves and oscillations; (ii) recollect the classical wave equation; (iii) revise the equations governing the motion of compressible and incompressible fluids; and (iv) acquaint ourselves with the process which gives us linear equations when disturbance amplitudes are small. The oscillatory nature of particle motions in waves in fluids and other material media also has to be emphasised.

L1 will cover such basic notions as what we mean by a wave, an oscillation and a (normal?) mode. Oscillatory systems, such as simple and coupled pendula, will be helpful here. If we think of waves in terms of travelling patterns of displacements of something, then notions of transverse, longitudinal and mixed waves arise naturally. We shall work through familiar definitions of amplitude, period, frequency, wavelength, wavenumber and phase velocity.

I have heard it seriously proposed that a course on waves should not spend time on the classical wave equation $c^2\nabla^2 F = \partial^2 F/\partial t^2$, and it will become clear during the course why this is not an entirely frivolous idea. However, L2 will recall the wave equation, give a few examples of its occurrence, and consider solutions (both general and specific); this will give us a pretext for some simple functional analysis. As an antidote, we'll look at a fascinating physical and mathematical situation: a diffusion problem in which oscillatory forcing is applied.

After recalling basic ideas of Fourier analysis, in L3 we'll discuss dispersive waves (whose phase speeds depend on wavelength) and non-dispersive waves. The distinctions between neutral, decaying and amplifying waves - each of which will be encountered later - will be introduced too. The dispersion equation which relates angular frequency ω and wavenumber k, $\omega = \omega(k)$, is fundamental here. We shall examine the concept of group velocity (for neutral waves), and consider the formula $c_g = d\omega/dk$.

In L4 and L5 the equations of motion, thermodynamics, continuity and state will be presented, with careful distinction made between the case of a (compressible) perfect gas and of an incompressible fluid. We shall derive the linear forms of these equations which describe the behaviour of small amplitude disturbances of some equilibrium state; these linear equations form the basis for most of the course. Eulerian and Lagrangian views in fluid dynamics will be defined and contrasted.

Waves in and on simple fluids (L6 - L15)

In L6 we examine the archetypal wave problem in fluid dynamics - waves on the surface of an incompressible liquid under gravity. The dispersion relation shows that short waves are dispersive, but long waves non-dispersive - they actually obey the classical wave equation. (The long wave limit is also known as the shallow water case.) Long waves are longitudinal in character. Short waves are mixed longitudinal/transverse in character; particles move in circles. The dispersion properties of short waves on water account qualitatively for the occurrence of particularly rough seas over shallows such as the Dogger Bank. Similar refraction and focusing properties will be noted in L7 regarding acoustic waves in a perfect gas having temperature inhomogeneities. Pure acoustic waves - in an isothermal gas in the absence of gravity - will also, of course, be discussed. These waves are non-dispersive and longitudinal; like long waves on water, they are quite easily rationalised in physical terms.

L8 deals with acoustic waves in the presence of gravity, but limits attention to an isothermal perfect gas atmosphere and to the cases of purely vertical motion and purely horizontal motion. The vertically propagating wave is slightly dispersive, but the one involving only horizontal motion is non-dispersive - it is the Lamb wave, which has an exponential transverse variation of amplitude and is known as an edge wave.

L9 reverts to incompressible fluids, but this time to stratified incompressible fluids (in which the mean density ρ varies with height z because of temperature inhomogeneities). An heuristic argument shows that particles of fluid displaced in the vertical oscillate freely with frequency N, where $N^2 = (g/\rho)d\rho/dz$. We note that the same argument gives $N\cos\alpha$ for the frequency of oscillation of particles constrained to move at an angle α to the vertical. Waves in a stratified incompressible fluid are known as internal gravity waves; they are dispersive transverse waves and have various distinctive properties.

The problem of uniform flow U of a stratified incompressible fluid over a wavy lower boundary (L10) allows us to tie the results of L9 together neatly. "Propagating" or "trapped" wave responses are possible, depending only on the parameter N/Uk, where k is the wavenumber of the boundary, and we can interpret this result in terms of the existence or non-existence of a free oscillation frequency Uk. This is a very satisfactory problem and result.

L11 deals with the heuristic particle oscillation argument as applied to a stably stratified perfect gas atmosphere. The frequency of vertical oscillations is N, given by $N^2 = (g/\theta)d\theta/dz$, θ being the potential temperature. It is a natural step to consider next the case of internal gravity

waves in an isothermal perfect gas atmosphere (L12). The trouble, of course, is that acoustic waves can occur too! We find a quartic dispersion relation, two roots of which correspond to a pair of acoustic waves, and the other two to a pair of internal gravity waves.

This situation leads us to consider (L13) how we might approximate the equations of motion so as to remove the acoustic waves, but leave the internal gravity waves (ideally) unaffected. There are two ways of doing this: by applying the hydrostatic approximation, and by applying an incompressibility approximation in the continuity equation. The hydrostatic approximation does not remove the Lamb wave, and changes gravity wave phase speeds noticeably. The incompressibility approximation removes all acoustic waves (including the Lamb wave) and leaves gravity wave phase speeds intact. We shall see also that you can't just go around dropping terms from the equations as you please; if approximations are not made carefully, equations with spurious energy sources or other undesirable features may result.

In L14 we discuss the effects of frame rotation on most of the waves we've looked at so far. The presence of Coriolis terms in the horizontal components of the momentum equation allows inertial oscillations at frequency f (= Coriolis parameter = 2 x local frame rotation rate), but there is no new species of wave motion so long as latitude variations of f are neglected. What happens is that the properties of the waves previously found are modified; in general, more dispersion is introduced. Shallow water waves become (dispersive) Poincaré waves; (dispersive) internal gravity waves become (even more dispersive) inertio-gravity waves. A new sort of edge wave, the Kelvin wave, occurs. The Kelvin wave is seriously anisotropic, exhibiting geostrophic balance transverse to its direction of propagation but behaving like a shallow water wave as regards motion in the direction of propagation. The Kelvin wave is partly responsible for the large tidal range found along the coast of northern France.

A simple case of flow instability will be studied in L15. We derive a dispersion relation, $\omega = \omega(k)$, and find that ω may be a complex number! This means that the amplitude of the corresponding wave either grows with time or decays with time. (In fact, growing and decaying waves almost always occur in pairs, as conjugate roots of the dispersion equation.) Both the Kelvin-Helmholtz or the Rayleigh-Taylor instability problems will be described, and one of them analysed in detail. It will be emphasised that instability problems are thought experiments whose results require careful interpretation.

More advanced topics (L16 – L20)

L16-L20 will cover in outline a number of topics which take us to the threshold of much recent and current research in meteorology and oceanography. During earlier lectures we will have largely arranged our problems so that the physical state upon which wave motion occurs is in a certain sense homogeneous. (In mathematical terms this means that key factors in our equations are constant coefficients rather than spatially varying functions.) For example, the "background" state will have been a state of rest or uniform flow, with uniform buoyancy frequency if there is any stratification at all. In L6 and L7 we will have discussed refraction and focusing of surface water waves and acoustic waves in qualitative physical terms. L16 will apply some basic mathematical apparatus to such problems of wave motion in inhomogeneous fluids. It is as if we were studying the propagation of light in a medium of spatially-varying refractive index - the mathematics is much the same. We shall discuss how the WKB method can formalise notions of ray tracing of surface water and internal gravity waves. The difficulties of "critical levels" - where the effective refractive index becomes infinite - and the importance of group velocity concepts will be brought out.

In L13 we will have noted the effects of uniform rotation on gravity and acoustic waves, assuming that the Coriolis parameter (f) is constant. Another type of wave motion, the Rossby wave, is found if the latitude variation of the Coriolis parameter is taken into account. The barotropic vorticity equation, the Rossby wave and barotropic instability will be examined in L17. Our main reason for looking at these phenomena is that the barotropic instability problem provides the simplest vehicle for a discussion (in L18 and L19) of the mathematics of instabilities in fluids. Linear dynamical systems characterised by self-adjoint differential operators (or symmetric matrices) have a complete orthogonal set of discrete modes, and much of classical physics is concerned with such systems. Fluid dynamics, however, frequently produces linear problems involving operators which are not self-adjoint and which possess an incomplete set of (possibly non-orthogonal) discrete modes. The barotropic instability problem will be discussed in this light, and it will be noted that the concept of pseudomomentum does much to restore the comfortable self-adjoint veneer in which dynamics tends to be sold during undergraduate courses.

Finally, L20 will examine the behaviour of large amplitude waves. In most situations, nonlinear effects be important at large wave amplitude, with transfer of energy between different space and time scales of motion. Sometimes, however, it turns out that the nonlinear terms vanish, so that a supposedly restricted wave solution applies also at large amplitude. Sometimes a balance occurs between nonlinearities and dispersion or viscosity; coherent structures called solitons may emerge and persist. In both cases, a new stability problem (regarding the large amplitude wave or soliton) is suggested. Shock waves, bores and wave breaking will be briefly discussed in the light of these issues.

WAVES IN THE ATMOSPHERE AND OCEAN (Module MT728)

A course of 20 lectures.

OBJECTIVE

To develop an understanding of a variety of geophysical wave phenomena, many of which are of considerable environmental importance.

CONTENT

Foundations (5 lectures)
Review of basic concepts and terminology: oscillations, (normal) modes and waves; amplitude, period, frequency, wavelength, wavenumber, phase velocity; transverse, longitudinal and mixed waves.
Classical wave equation: occurrence and general solution. Wavy solutions of the diffusion equation.
Fourier analysis. Non-dispersive and dispersive waves. Neutral, decaying and amplifying waves. Group velocity.
The equations of fluid motion: perfect gas, incompressible fluids. Small amplitude approximations. Eulerian and Lagrangian views.

Waves in and on simple fluids (10 lectures)
Surface waves on water: 'deep' (dispersive) and 'shallow' (non-dispersive) limits.
Pure acoustic waves; refraction and focusing.
Acoustic waves in the presence of gravity: horizontal and vertical propagation in an isothermal atmosphere; Lamb waves.
Buoyancy oscillations and internal gravity waves in stratified, incompressible fluids; dispersion properties.
Flow over wavy boundaries: trapped and propagating wave responses.
Buoyancy oscillations in a perfect gas atmosphere.
Mixed acoustic/gravity waves in an isothermal, perfect gas atmosphere.
Approximations and filtering; consistently soundproofing the equations of motion of a perfect gas.
Effects of uniform rotation: inertial oscillations; Poincaré, Kelvin and inertio-gravity waves; geostrophic balance.
Kelvin-Helmholtz and Rayleigh-Taylor instability.

More advanced topics (5 lectures)
Some of the following will be discussed, though only in overview terms
Waves in inhomogeneous fluids: ray tracing and the WKBJLG approximation; critical levels.
Rossby waves. Barotropic vorticity equation. Barotropic instability.
Mathematics of instabilities in fluids: modal and non-modal disturbances; self-adjoint and non-self-adjoint operators; completeness and incompleteness; pseudomomentum.
Large amplitude waves: linearity; weak non-linearity; soliton behaviour; transfer of energy between different space and time scales; shock waves and bores; stability of large amplitude waves; wave breaking.

ASSESSMENT

By a University examination taken during Term 9. Students are asked to answer two out of three questions in 2 hours.

REFERENCES

Acheson, D. J.	1990	*Elementary Fluid Dynamics.* Clarendon Press
Coulson, C. A.	1961	*Waves.* Oliver and Boyd
Drazin, P. G. and Reid, W. H.	1981	*Hydrodynamic Stability.* Cambridge University Press
Gill, A. E.	1982	*Atmosphere-Ocean Dynamics.* Academic Press
Holton, J. R.	1992	*An Introduction to Dynamic Meteorology.* Academic Press
Lighthill, M. J.	1996	*Waves in Fluids.* Cambridge University Press
Longair, M. S.	1984	*Theoretical Concepts in Physics.* Cambridge University Press
Pedlosky, J.	1987	*Geophysical Fluid Dynamics.* Springer-Verlag
Turner, J. S.	1973	*Buoyancy Effects in Fluids.* Cambridge University Press

Gill's book is particularly recommended, and covers much more than just waves. Lighthill's book is more specific, and in some respects more advanced. The other texts may be useful for consolidation in particular subject areas.

Waves in the Atmosphere and Oceans

1. Review of basic concepts and terminology

Waves in fluids involve the motion of elements, or *parcels*, of fluid. As in other media, *longitudinal waves* involve parcel displacements parallel to the direction of travel of the wave; pure acoustic waves are of this type. *Transverse waves* involve displacements perpendicular to the direction of travel of the wave; internal gravity waves are examples. Many waves are of mixed type, involving parcel displacements both parallel and perpendicular to the direction of travel of the wave; surface waves on deep water belong to this class.

Waves need not be sinusoidal, or even periodic, but it's often helpful to examine sinusoidal waves. Plane, sinusoidal waves moving in the x direction may be represented by a variety of mathematical forms:

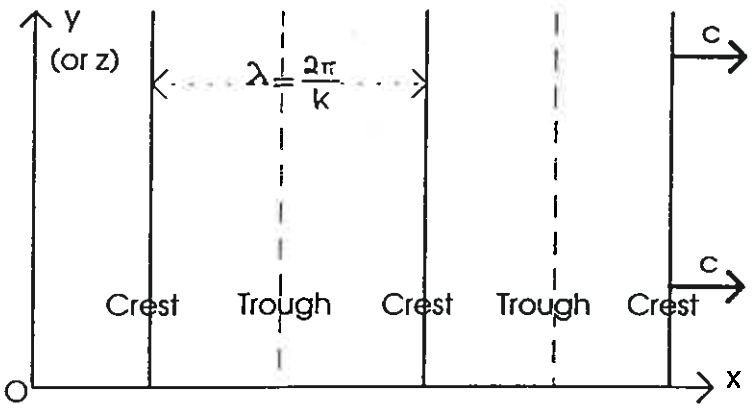
$$a\begin{matrix}\sin\\\cos\end{matrix}(kx-\omega t),\quad a\begin{matrix}\sin\\\cos\end{matrix}(k[x-ct]),\quad A\exp(i[kx-\omega t]),\quad A\exp(ik[x-ct]) \qquad (1)$$

Here $a\begin{matrix}\sin\\\cos\end{matrix}(\theta(x,t))$ means "either $a\sin\theta$ or $a\cos\theta$", or - more generally - "any linear combination of $\sin\theta$ and $\cos\theta$ having total amplitude a"; an equivalent form in the second case is $a\sin(\theta+\phi)$, where ϕ is some constant phase factor. When the complex exponential form is used it will be assumed that the real part represents the function that's of interest; the amplitude A is a complex number, in general. In (1):

k is the wavenumber, related to the wavelength λ as $k = 2\pi/\lambda$;

ω is the frequency, related to the wave period τ as $\omega = 2\pi/\tau$.

Strictly, k and ω are *angular* wavenumber and *angular* frequency, with units radians per unit length and radians per unit time respectively. Wavenumber is sometimes defined as $1/\lambda$ (number of wavelengths per unit length) and frequency as $1/\tau$ (number of wave periods per unit time).



The phase velocity, c, is the rate at which wave crests or troughs move in the x direction. It is related to the frequency by $\omega = ck$. The wave moves to the right (x increasing) or to the left (x decreasing) according as c and ω are positive or negative.

[According to another convention, ω is counted as essentially positive, and the sign of k determines the direction of movement.]

Our analysis in this course will often lead to an equation for ω or c in terms of the wavenumber k. This equation is known as the **dispersion relation**. If c is independent of k, then the phase speed of the wave is independent of wavelength; a disturbance of arbitrary initial shape will move at the phase speed *without change of shape*. This is the **non-dispersive** case. [Notice that the frequency varies with wavenumber in the non-dispersive case, since $\omega = ck$.] If c depends on k, then an arbitrary initial disturbance will *disperse* in the sense that its shape will change with time, since its different Fourier components will travel at different speeds. This is the **dispersive** case. In so far as a shape can still be defined, the group velocity

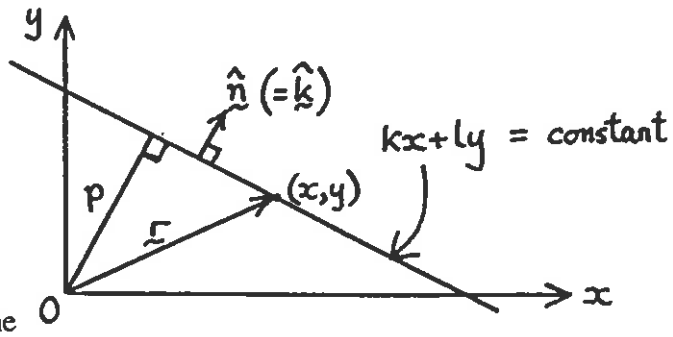
$$c_g = \frac{d\omega}{dk}$$

gives its speed (and direction). Notice that non-dispersive waves have equal group velocity and phase velocity. Group velocity will be discussed more in a later lecture. Returning to (1), two further aspects are of interest:

a) *Extensions to 2 and 3 space dimensions are fairly straightforward.* In 2D, plane sinusoidal waves have the form

$$a\begin{matrix}\sin\\ \cos\end{matrix}(kx+ly-\omega t), \quad A\exp(i(kx+ly-\omega t))$$

Here k and l are components of a *wave vector* $\mathbf{k}$, whose magnitude is $K = \sqrt{k^2+l^2}$: $\mathbf{k} = (k, l)$. Note that $kx + ly = \mathbf{k}.\mathbf{r}$, where $\mathbf{r} = (x,y)$. Since the equation of a line having unit normal $\hat{\mathbf{n}}$ is $\mathbf{r}.\hat{\mathbf{n}} = p$, where p = perpendicular distance of the line from the origin (see diagram), it follows that $\mathbf{k}$ is normal to the lines of constant phase of the wave at any instant. In other words, $\mathbf{k}$ points in the direction of travel of the wave. In terms of the unit vector $\hat{\mathbf{k}} = \mathbf{k}/K$, the phase $(kx+ly-\omega t)$ can be written as $K(\hat{\mathbf{k}}.\mathbf{r} - ct)$, giving forms alternative to those given above. The 3D case ($\mathbf{k} = (k, l, m)$, $K = \sqrt{k^2+l^2+m^2}$, $\mathbf{r} = (x, y, z)$) is similar.



b) *The forms involving ω emphasize the oscillatory, or modal, aspects of waves; the forms involving c emphasize the propagatory aspects of waves.* Waves in material media such as fluids always have oscillatory and propagatory aspects. To recap on oscillations in particle systems we may consider: (i) a simple pendulum [which really is simple]; (ii) coupled pendula of equal mass [a system possessing 2 orthogonal - or "normal" - modes of oscillation]; and (iii) coupled pendula of unequal mass [whose 2 modes are not orthogonal in the usual sense - though they are orthogonal in a certain scaled coordinate system]. In each of these problems we make an assumption of *small amplitude* oscillations in order to obtain *linear* differential equations; the problems also give us a perspective which will be useful later on in the course.

2. The classical wave equation

The wave equation in 3 space dimensions is $c^2\nabla^2 F = \partial^2 F/\partial t^2$,

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ in Cartesian (x, y, z) terms.

The 2D and 1D versions are obvious specializations. Apart from its (occasional!) occurrence in fluid dynamics, the wave equation arises in many physical contexts, often as a result of a small amplitude approximation [SAA]. Examples are:

String under tension: $c = \sqrt{T/\rho}$ in this 1D case (T = tension, ρ = mass per unit length). SAA is involved and the waves are transverse.

Bars and springs: $c = \sqrt{\lambda/\rho}$ in this 1D case, where now λ is a Hooke's Law constant. SAA is involved and the waves are longitudinal.

Membranes under tension: $c = \sqrt{T/\rho}$ in this 2D case, where now T = tension per unit length, and ρ = mass per unit area. SAA; transverse waves.

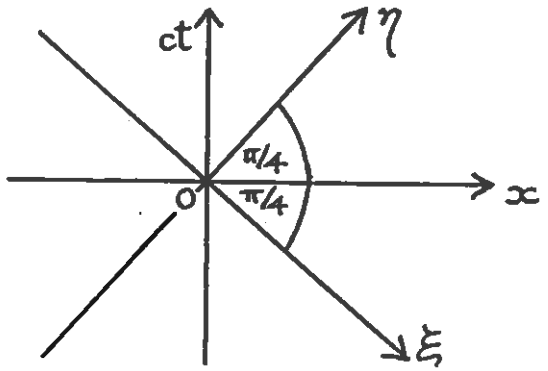
Electromagnetic waves: c = velocity of light. 3D, transverse waves; no SAA. Fluctuating quantities are electric and magnetic fields (and potentials).

The case of a string under (uniform) tension provides another example of modes of oscillation whose spatial forms are orthogonal to one another; i.e. ***normal modes.***

General solution (1D) Put $\xi = x - ct$ and $\eta = x + ct$ into $c^2\,\partial^2 F/\partial x^2 = \partial^2 F/\partial t^2$.

Using $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial \xi} + \dfrac{\partial}{\partial \eta}$

and $\dfrac{1}{c}\dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial \eta} - \dfrac{\partial}{\partial \xi}$,



we find $\dfrac{\partial^2 F}{\partial \xi \partial \eta} = 0$. This equation has the general solution $F(\xi,\eta) = G(\xi) + H(\eta)$, where G and H are *any* functions we can differentiate twice w.r.t. their arguments. Hence, in terms of x and t, the general solution is $F(x,t) = G(x-ct) + H(x+ct)$. [Notice that the sinusoidal wave forms we looked at in L1 are of this type.] Extension to 2D and 3D is reasonably straightforward; plane waves result.

D'Alembert's solution in 1D Suppose that we are given $F(x,0)$ $[= \Phi(x)]$ and $\partial F/\partial t\,(x,0)$ $[= \theta(x)]$. (These are *Cauchy conditions*, which are appropriate for a hyperbolic p.d.e. such as the wave equation.) From $F(x,t) = G(x-ct) + H(x+ct)$ we have simply

$$G(x) + H(x) = \Phi(x) \quad . \qquad (1)$$

Now $$\frac{\partial F}{\partial t} = -cG'(x-ct) + cH'(x+ct)\ ,$$

where the primes mean differentiation w.r.t. the arguments. Hence

$$cH'(x) - cG'(x) = \theta \ .$$

Integration gives $$H(x) - G(x) = \frac{1}{c}\int_a^x \theta(x')dx' \qquad (2)$$

where a is an arbitrary constant. From (1) and (2) we can solve for H(x) and G(x), then write $H(x+ct)$ and $G(x-ct)$ and finally construct $F(x,t)$; the arbitrary constant cancels, and we find d'Alembert's solution of the wave equation:

$$2F(x,t) = \Phi(x-ct) + \Phi(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} \theta(x')dx'$$

Only the conditions within $\pm ct$ of point x are relevant to the solution at x at time t.

Spherical progressive waves The 3D wave equation, with spherical symmetry, is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}$$

Now (rather amazingly) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rF)$, so our 3D wave equation (with spherical symmetry) can be written as $\frac{\partial^2}{\partial r^2}(rF) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rF)$.

Hence the general solution is $F(r,t) = \frac{1}{r}G(r-ct) + \frac{1}{r}H(r+ct)$.

If you think this is a trivial result, try doing the 2D problem with central symmetry!

Wavy (?) solutions of the diffusion equation

Frequently in this course we shall search for wave-like solutions to equations other than the classical wave equation. As an extreme example of this procedure, we consider an application to the *diffusion equation.*

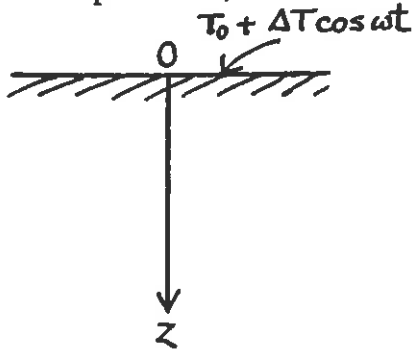
Suppose that we wish to solve $\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial z^2}$,

given that the temperature T must satisfy

$T(0,t) = T_0 + \Delta T\cos\omega t$; $T(\infty,t)$ finite.

The wavy trial solution $T = T_0 + \Delta T\exp[i(kz-\omega t)]$ (real part only) obeys the boundary conditions, so let's substitute it into the diffusion equation. We find

$$i\omega = \kappa k^2 .$$



Since ω is real (it's imposed) k must be complex. Using $(1+i)^2 = 2i$ we find

$$k = \pm(1+i)\mu \quad \text{with} \quad \mu^2 = \omega/2\kappa .$$

Discarding a possible solution having exponential increase with z, we find

$$T = T_0 + \Delta T\exp(-\mu z)\cos(\mu z - \omega t).$$

You can readily check that this satisfies the diffusion equation and the boundary conditions. The quantity $1/\mu$ is called the damping depth; it is the depth at which the amplitude of the temperature variation has decreased to $1/e$ of the value at $z = 0$. π/μ is the depth at which the temperature variation is out of phase with the surface variation. This problem gives us an example of a *complex dispersion relation.* In later examples it will usually be the frequency that is complex.

3. Dispersion relations; neutral, amplifying and decaying waves; group velocity

The *dispersion relation* gives the frequency ω of sinusoidal waves in terms of wavenumber k: $\omega = \omega(k)$ in $\exp\{i(kx - \omega t)\}$. For a diffusion problem we found in L2 the dispersion relation $i\omega = \kappa k^2$, implying a complex wavenumber (and a spatially-decaying wave) because the frequency was real. In many problems, the wavenumber k must be real, and a complex frequency is implied in some cases. Suppose that $\omega(k) = A(k) + iB(k)$, where A, B are real. Then:

$$\exp\{i(kx - \omega t)\} \;=\; \exp\{i(kx - At - iBt)\} \;=\; \exp\{i(kx - At)\}\exp\{Bt\}.$$

So:
- if $B > 0$, the amplitude increases exponentially with time - the wave *amplifies*;
- if $B = 0$, the amplitude is constant - the wave is *neutral*;
- if $B < 0$, the amplitude decreases exponentially with time - the wave *decays*.

When an amplifying wave occurs, a decaying wave is nearly always possible too, and vice versa. As an example, consider the hypothetical dispersion relation $\omega^2 = k_0^2 - k^2$; clearly, for $k > k_0$, $\omega = \pm i\lambda$, with $\lambda^2 = k^2 - k_0^2$.

Group velocity (for *neutral* waves)
We discussed in L1 the distinction between dispersive waves [$\omega/k \neq d\omega/dk$] and non-dispersive waves [$\omega/k = d\omega/dk$]. Here we interpret the *group velocity*, $c_g = d\omega/dk$.

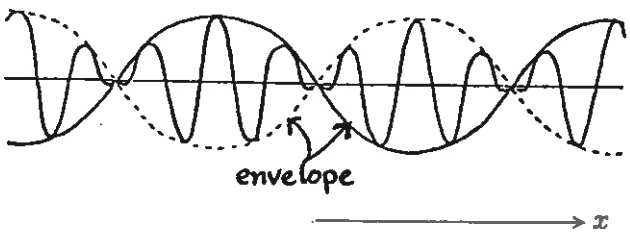
a. *Superposition of two sinusoidal waves having different wavenumbers and frequencies*
A convenient general representation of two waves having equal amplitude is

$$F(x,t) \;=\; \cos[(k + \Delta k)x - (\omega + \Delta\omega)t + \phi] \;+\; \cos[(k - \Delta k)x - (\omega - \Delta\omega)t - \phi] \qquad (1)$$

The phase functions in (1) can be re-written $[(kx - \omega t) \pm (x\Delta k - t\Delta\omega + \phi)]$, so we find

$$F(x,t) = 2\cos\left\{\Delta k\left(x - \frac{\Delta\omega}{\Delta k}t\right) + \phi\right\}\cos[kx - \omega t]$$

This general result shows that, if $\Delta\omega$ and Δk are small, $\cos[kx - \omega t]$ is modulated by a slowly-varying envelope which moves in the x direction at the *group velocity* $c_g = d\omega/dk \cong \Delta\omega/\Delta k$.



Notice that the two component waves in (1) will have constant phase difference in a frame of reference moving such that $x\Delta k - t\Delta\omega = \phi/2$. This frame moves in the x direction with the velocity $dx/dt = \Delta\omega/\Delta k \cong d\omega/dk = c_g$.

b. *Movement of the envelope of a wave packet*
Most small-amplitude disturbances can be represented by a Fourier integral as

$$F(x,t) \;=\; \int_{-\infty}^{+\infty} a(k)\exp[i(kx - \omega t)]dk$$

Suppose that $a(k)$ is quite tightly peaked around $k = k_0$. Then we can expand the phase factor about k_0 and $\omega_0 = \omega(k_0)$ and obtain a useful approximation from the first few terms:

$$kx - \omega t \;=\; k_0 x - \omega_0 t + (k - k_0)x - (d\omega/dk)\big|_{k_0}(k - k_0)t - (d^2\omega/dk^2)\big|_{k_0}(k - k_0)^2 t\ /2 - \ldots$$
$$\cong\; k_0 x - \omega_0 t \;+\; (k - k_0)(x - c_g t)$$

Hence $$F(x,t) \equiv \exp\left[i(k_0 x - \omega_0 t)\right] \int_{-\infty}^{+\infty} a(k) \exp\left[i(k-k_0)(x - c_g t)\right] dk \qquad (2)$$

The integral in (2) is a function of $(x - c_g t)$ and k_0 only. Once again we obtain the result that a wave train is modulated by an envelope that moves with the group velocity. The result (2), and the reasoning leading to it, are made clearer by considering particular choices of $a(k)$.

For example, taking $a(k) = \exp\left[-\sigma(k-k_0)^2/2\right]$ gives the familiar *Gaussian* wave-packet.

At $t = 0$,

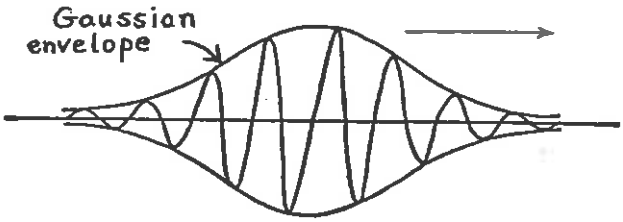
$$F(x,0) = \int_{-\infty}^{+\infty} \exp\left[-\sigma(k-k_0)^2/2 + ikx\right] dk = \exp[ik_0 x] \int_{-\infty}^{+\infty} \exp\left[-\sigma\kappa^2/2 + i\kappa x\right] d\kappa \qquad (3)$$

(where $\kappa = (k - k_0)$).

By writing $\sigma\kappa^2 - 2i\kappa x = \sigma\left[(\kappa - ix/\sigma)^2\right] + x^2/\sigma$ and using $\int_{-\infty}^{+\infty} \exp(-by^2) dy = \sqrt{\pi/b}$,

it is reasonably straightforward to show that

$$F(x,0) = \sqrt{\frac{2\pi}{\sigma}} \exp\left\{-\frac{x^2}{2\sigma}\right\} \exp[ik_0 x]$$



To find $F(x,t)$, use (2). Expansion of ω about k_0, and manipulation as before, leads to

$$F(x,t) = \sqrt{\frac{2\pi}{\sigma}} \exp\left\{-\frac{(x - c_g t)^2}{2\sigma}\right\} \exp\left[i(kx_0 - \omega_0 t)\right]$$

It is possible to include the $d^2\omega/dk^2$ term in the Taylor expansion of $\omega(k)$ about k_0. The major effect is that the envelope widens slowly with time, and its maximum value decreases.

c. *Local phase argument*

Suppose that a wave field may be expressed as $F(x,t) = A(x,t)\exp(i\theta(x,t))$, where A and θ are amplitude and phase functions. Local wavenumber and frequency may be defined as $k = \partial\theta/\partial x$, $\omega = -\partial\theta/\partial t$ (which give familiar results when $\theta = kx - \omega t$). Then, clearly:

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \Rightarrow \frac{\partial k}{\partial t} + \frac{d\omega}{dk}\frac{\partial k}{\partial x} = 0 \quad \text{i.e.} \quad \frac{\partial k}{\partial t} + c_g \frac{\partial k}{\partial x} = 0$$

Hence $k(x,t)$ *appears constant to an observer moving with the group velocity.*

d. *Energy propagation*

From **a** and (especially) **b**, it is clear that the "signal" or the "energy" moves with the group speed rather than the phase speed. (Of course, in non-dispersive systems the phase speed and the group speed are the same anyway.) This important aspect of group velocity is apparent in particular cases - such as surface waves on deep water.

e. *Extensions to 2D and 3D*

For 2D waves of the form $A\exp\left[i(kx + ly - \omega t)\right]$, with dispersion relation $\omega = \omega(k,l)$, the group velocity is the vector $\mathbf{c}_g = (\partial\omega/\partial k, \partial\omega/\partial l)$. Similarly, for 3D waves of the form $A\exp\left[i(kx + ly + mz - \omega t)\right]$, with dispersion relation $\omega = \omega(k,l,m)$, the group velocity is the vector $\mathbf{c}_g = (\partial\omega/\partial k, \partial\omega/\partial l, \partial\omega/\partial m)$. It is a useful *exercise* to derive these 2D and 3D generalisations by applying arguments **a** and **c**, above; this is much easier than you might expect, and shows our 1D results in a new perspective.

4. The equations of fluid motion: perfect gas and incompressible fluid

Having revised some basic wave concepts in L1-3 we now review the equations of motion.

Total and local time derivatives; advection

Suppose $\mathfrak{I} = \mathfrak{I}(\mathbf{r}, t)$ is some meteorological field. $\mathfrak{I}$ might be a scalar quantity - such as temperature, or a vector - such as the flow velocity $\mathbf{u}$. First-order Taylor expansion of $\mathfrak{I}$ about $\mathfrak{I}(\mathbf{r}, t)$ gives

$$\delta\mathfrak{I} = \mathfrak{I}(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - \mathfrak{I}(\mathbf{r}, t) = (\delta\mathbf{r}.\nabla)\mathfrak{I} + (\partial\mathfrak{I}/\partial t)\delta t \quad . \qquad (1)$$

Eq (1) applies to any (infinitesimal) choice of $\delta\mathbf{r}$, δt. Choosing $\delta t \neq 0$, we obtain the change in $\mathfrak{I}$ from $\mathfrak{I}(\mathbf{r}, t)$ over the interval $[t, t + \delta t]$ for any choice of $\delta\mathbf{r}$ (whatever its direction). In the *special case* in which $\delta\mathbf{r}$ is chosen to be the displacement in time δt achieved by the flow velocity $\mathbf{u}$ at position $\mathbf{r}$, we have $\delta\mathbf{r}/\delta t = \mathbf{u}$, and (1) then becomes

$$\frac{D\mathfrak{I}}{Dt} \equiv \frac{\delta\mathfrak{I}}{\delta t} = (\mathbf{u}.\nabla)\mathfrak{I} + \frac{\partial\mathfrak{I}}{\partial t} \quad . \qquad (2)$$

$D\mathfrak{I}/Dt$ is the rate of change of $\mathfrak{I}$ following a parcel of air; it is known as the *total* (or material, or substantial, or Lagrangian) time derivative of $\mathfrak{I}$. $\partial\mathfrak{I}/\partial t$ is the *local* (or Eulerian) time derivative of $\mathfrak{I}$; it is the rate of change of $\mathfrak{I}$ at a point fixed in our chosen coordinate frame. The quantity $-(\mathbf{u}.\nabla)\mathfrak{I}$ is called the advection of $\mathfrak{I}$.

First law of thermodynamics (T1)

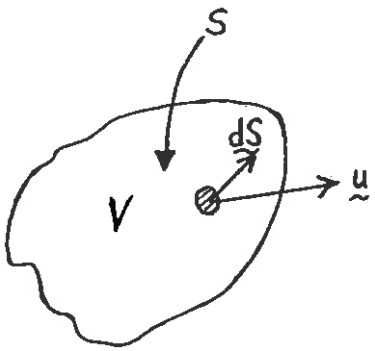
Suppose that a parcel of air having current temperature T and specific volume α is subject to heating δH per unit mass. According to T1, the associated changes of δT and $\delta\alpha$ are related by

$$c_v\delta T + p\delta\alpha = T\delta S = \delta H \quad . \qquad (3)$$

Here c_v is the specific heat at constant volume, p is the pressure of the parcel of air, and δS is the change in entropy. Since T1 applies to the element of air as it moves, and $\alpha = 1/\rho$, [ρ = density], (3) implies that

$$c_v\frac{DT}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = Q \qquad (4)$$

where Q $(= T\,DS/Dt)$ is the heating rate per unit mass.



Mass continuity

The mass budget of a fixed ("control") volume V bounded by a fixed surface S may be expressed as

$$\frac{\partial}{\partial t}\int_V \rho dV = -\int_S \rho\mathbf{u}\cdot\mathbf{dS} = -\int_V \nabla\cdot(\rho\mathbf{u})dV \qquad (5)$$

by the divergence theorem. Eq (5) applies to any control volume V, so the local equality

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \qquad (6)$$

must hold. Equation (6) is the Eulerian form of the (mass) continuity equation. An alternative version (obtained using (2), with $\mathfrak{I} = \rho$), is the Lagrangian form

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0 \quad . \qquad (7)$$

Perfect gas law $p = \rho RT$ (8)

Here R is the gas constant per unit mass. Eq (8) has no time derivatives. In meteorological parlance, it is a *diagnostic* equation. Equations involving time derivatives are called *prognostic*.

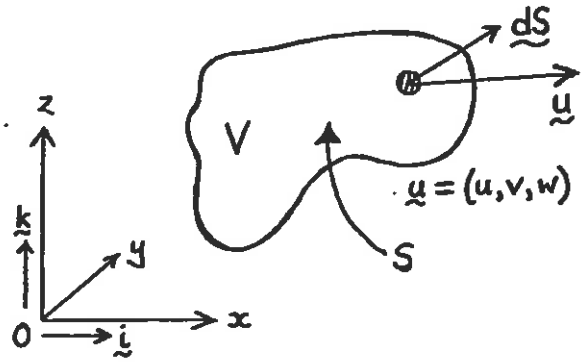
Newton's second law of motion (N2)

Consider the budget of x momentum in a control volume V. We have to allow for the flux of x momentum through its boundary S as well as for the rate of change of x momentum achieved by the forces acting at S and throughout V. If we neglect viscosity then the only boundary force is the pressure, p. Suppose that the force per unit mass acting within the volume is $\mathbf{F}$. Then:

$$\frac{\partial}{\partial t}\int_V \rho u dV = -\int_S \rho u \mathbf{u}\cdot d\mathbf{S} - \int_S p\mathbf{i}\cdot d\mathbf{S} + \int_V \rho F_x dV = \int_V \{-\nabla\cdot(\rho u\mathbf{u} + p\mathbf{i}) + \rho F_x\} dV \quad (9)$$

Here $\mathbf{i}$ = unit vector in the x direction (see diagram). Noting that $\nabla\cdot(\phi\mathbf{a}) = \phi\nabla\cdot\mathbf{a} + \mathbf{a}\cdot\nabla\phi$ for any scalar ϕ and vector $\mathbf{a}$, that $\nabla\cdot\mathbf{i} = 0$, and that V is arbitrary, we obtain from (9) the local equality

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + u\nabla\cdot(\rho\mathbf{u}) + \rho\mathbf{u}\cdot\nabla u + \frac{\partial p}{\partial x} = \rho F_x .$$



The continuity equation (6) shows that the second and third l.h.s. terms cancel. So we obtain simply

$$\frac{\partial u}{\partial t} + \mathbf{u}\cdot\nabla u + \frac{1}{\rho}\frac{\partial p}{\partial x} = F_x ,$$

or (using (2)) $$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} = F_x .$$

We can look at the y and z components of the momentum budget in the same way, and obtain similar results. The vector form of the momentum equation is then seen to be

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p = \mathbf{F} . \quad (10)$$

This treatment assumes that our *Oxyz* coordinate system is not rotating. If *Oxyz* is rotating about the local vertical with angular velocity $\Omega\mathbf{k}$, ($\mathbf{k}$ = unit vector in the z direction) then Coriolis and centrifugal terms appear in (10). If $\mathbf{F}$ consists *only* of apparent gravity, $-g\mathbf{k}$, (10) becomes

$$\frac{D\mathbf{u}}{Dt} + 2\Omega\mathbf{k}\times\mathbf{u} + g\mathbf{k} + \frac{1}{\rho}\nabla p = 0 \quad (11)$$

[See L14.] Given suitable boundary conditions, (4), (7), (8) and (10) or (11) [with (2)] determine the time evolution of the flow.

Incompressible fluid

For liquids, compressibility is often unimportant, and a useful idealisation is an equation of state which relates density to temperature alone:

$$\rho = \bar{\rho}\left[1 - a(T - \bar{T})\right] \quad (12)$$

in which $\bar{\rho}$, $\bar{T}$ and a are constants. This may be used in conjunction with:

an adiabatic form of the thermodynamic equation: $$\frac{DT}{Dt} = 0 ; \quad (13)$$

an *incompressible* form of the continuity equation: $$\nabla\cdot\mathbf{u} = 0 ; \quad (14)$$

a *Boussinesq* form of N2: $$\frac{D\mathbf{u}}{Dt} + 2\Omega\mathbf{k}\times\mathbf{u} + g\frac{(\rho - \bar{\rho})}{\bar{\rho}}\mathbf{k} + \nabla\left(\frac{(p - \bar{p})}{\bar{\rho}}\right) = 0 \quad (15)$$

In (15), $\bar{\rho}g + \partial\bar{p}/\partial z = 0$. We could represent heating (in(13)) and viscosity (in (15)) if we wished. Variants of this simple incompressible fluid model will be encountered later on.

5. Further aspects of the equations of fluid motion; linearisation

We saw in L4 how N2, T1, the continuity equation and an equation of state [6 equations in all] determine the flow evolution, so long as we know the heating and the forces acting and we specify appropriate boundary conditions. For a perfect gas, the 5 variables T, ρ, $\mathbf{u}$ [=(u, v, w)] can be forecast using T1, continuity and N2; then p can then be diagnosed algebraically using the perfect gas law. Things are a little more complicated for an incompressible fluid, since the diagnosis of p requires a 3D Poisson equation to be solved.

As you might expect with a system of equations having so many dependent variables, there are many possible alternative and consequent forms. We note some of these in the case of a perfect gas.

T1 in terms of potential temperature (θ)

We derived T1 in L4 as

$$c_v \frac{DT}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = Q \qquad (1)$$

This can be cast into many different forms. By using the continuity equation we find

$$c_v \frac{DT}{Dt} + \frac{p}{\rho}\nabla\cdot\mathbf{u} = Q \qquad (1A)$$

Use of $p = \rho RT$ and the relation $c_p - c_v = R$ in (1) gives another favourite:

$$c_p \frac{DT}{Dt} - \frac{RT}{p}\frac{Dp}{Dt} = Q \qquad (1B)$$

A very useful form arises if we write (1B) in terms of potential temperature, θ:

$$\theta = T\left(p_{ref}/p\right)^{R/c_p} \quad \Rightarrow \quad \frac{D\theta}{Dt} = \frac{\theta}{Tc_p}Q \qquad (1C)$$

θ is the temperature a parcel of air would have if brought adiabatically and reversibly to pressure p_{ref} (usually chosen as 1000hPa). θ is related to the entropy per unit mass, S, by $\ln\theta = S/c_p$.

Exercise: Express the N2 pressure gradient term, $(1/\rho)\nabla p$, in terms of T and S.

Other conservation laws, other quantities, other coordinate systems .....

From the governing equations, conservation laws for other quantities can be derived.

- Taking the scalar product of N2 with $\mathbf{u}$ gives an equation for $D/Dt(\mathbf{u}^2/2)$, and is the first step towards a range of energy conservation laws.
- Taking the curl ($\nabla\times$) of N2 leads to an equation for $D\zeta/Dt$, where ζ is the (absolute) vorticity; this is the road to potential vorticity $P = (\zeta.\nabla\theta)/\rho$.
- Taking the divergence ($\nabla.$) of N2 leads to an equation for the 3-dimensional divergence $\nabla\cdot\mathbf{u}$; this option is rarely explored in meteorology (but it is a key step in the dynamics of incompressible fluids – for which $\nabla\cdot\mathbf{u} = 0$).

Exercise: What do you get if you cross N2 with the velocity – i.e. form $\mathbf{u}\times D\mathbf{u}/Dt$?

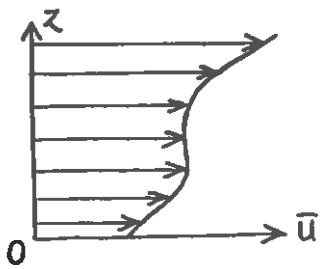
We haven't time to go into these things now, but will consider them when necessary.

Another aspect of meteorology worth noting is that vertical coordinates other than height z are often used (especially with the hydrostatic approximation): p, $\ln p$, p^{R/c_p}, $\sigma = p/p_{surface}$, θ, and other quantities may be found. We'll stick to z in this course.

Linearisation

A glance at the equations of motion shows that they contain many product terms, some of them (from $(\mathbf{u}.\nabla)\mathbf{u}$ in the vector form of N2) involving products of the same variable. The equations are *nonlinear*. For the next 10 lectures, at least, we're going to avoid this difficulty by working with *linearised* forms of the equations of motion. They arise like this. There will be some simple *basic state* - perhaps just a state of static equilibrium, with no motion - and on this state we shall impose small perturbations; the smallness of the perturbations justifies the neglect of terms involving products of perturbation quantities, and we obtain linear partial differential equations. These linear PDEs express the relationships that exist between the perturbation quantities. Assuming sinusoidal, wave-like solutions we derive a dispersion relation between their wavenumber and frequency. The whole procedure (which is a thought experiment, in effect) is similar in principle to the one used in the various non-fluid examples we discussed in earlier lectures.

As an example of the linearisation process, consider the fate of *Du/Dt* when the basic state consists simply of a flow in the x direction which is steady and varies only with z : $u = \bar{u}(z)$. Noting that the basic state does not vary with t, x or y, we write



$\mathbf{u} = \bar{u}(z)\mathbf{i} + (u', v', w')$ and expand *Du/Dt* as

$$\frac{Du}{Dt} = \frac{\partial u'}{\partial t} + (\bar{u}+u')\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\left(\frac{d\bar{u}}{dz} + \frac{\partial u'}{\partial z}\right).$$

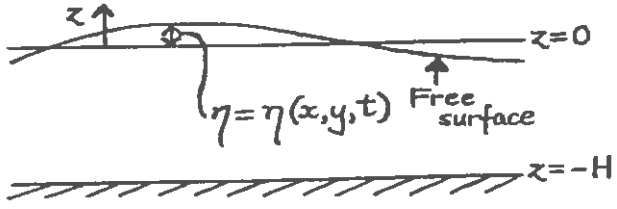
If we neglect terms which are products in the perturbation quantities, this reduces to the linear form

$$\frac{Du}{Dt} = \frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + w'\frac{d\bar{u}}{dz}$$

One more term drops out if $\bar{u}$ is independent of z; and if $\bar{u} = 0$ (basic state = state of rest) then we are just left with $\dfrac{Du}{Dt} = \dfrac{\partial u'}{\partial t}$

Linearisation for the surface wave problem

We apply this linearisation procedure to the problem of waves on the surface of an incompressible, non-rotating, homogeneous fluid (i.e. no variations of temperature or density). The basic state is a state of rest, with the fluid's free surface at $z = 0$; there is a rigid lower boundary at $z = -H$.



Eqs (12), (14), (15) of L4 become:

$\rho = \bar{\rho}$ (2); $\nabla\cdot\mathbf{u} = 0$ (3);

$$\frac{D\mathbf{u}}{Dt} + \nabla\left(\frac{p'}{\bar{\rho}}\right) = 0 \qquad (4);$$

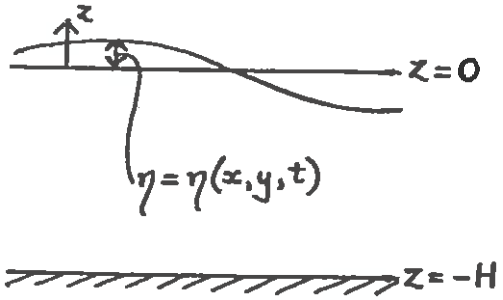
with $p = \bar{p} + p'$ and $\bar{p} = -\bar{\rho}gz$. This choice of $\bar{p}$ ensures that the basic state is in hydrostatic balance. The linearised form of N2 is simply $\dfrac{\partial \mathbf{u}'}{\partial t} + \nabla\left(\dfrac{p'}{\bar{\rho}}\right) = 0$ (5)

Hence, applying $\nabla\cdot\mathbf{u}' = 0$ we find $\nabla^2 p' = 0$ (6)

The perturbation pressure must obey Laplace's equation (a special case of Poisson's equation) in 3D. In L6 we shall consider how the boundary conditions at the surface and at $z = -H$, appropriately linearised, enable us to derive a dispersion relation for sinusoidal waves of surface displacement $\eta = \eta(x, y, t)$.

6. Surface waves on water: dispersion relation; deep and shallow limits

We saw in L5 that the pressure perturbation in the interior of the liquid must satisfy Laplace's equation ($\nabla^2 p' = 0$). All we have to do now is apply suitable boundary conditions. At $z = -H$ we require $w' = 0$; so, from (5) of L5, $\partial p'/\partial z = 0$ at $z = -H$. (1)



At the surface, we require $p = \bar{p} + p' = 0$; thus

$$p' = \bar{\rho} g \eta \quad \text{at} \quad z = \eta \qquad (2^*)$$

A third condition is that particles on the surface should remain on the surface: $D(z-\eta)/Dt = 0$ at $z = \eta$. The linearised form of this condition is

$$w' = \partial\eta/\partial t \quad \text{at} \quad z = \eta \qquad (3^*)$$

In fact, in our linearisation, it is sufficient to apply (3*) at $z = 0$ rather than $z = \eta$, and for good measure we apply (2*) at $z = 0$ too. Hence, our conditions are (1) and

$$p' = \bar{\rho} g \eta \quad \text{at} \quad z = 0 \qquad (2); \qquad w' = \partial\eta/\partial t \quad \text{at} \quad z = 0. \qquad (3)$$

Notice that we have 3 conditions to apply to a second order p.d.e.; the apparent over-specification leads to a dispersion relation. We have now formulated the problem.

Solution

We seek wavy solutions of the form

$$\eta = h\cos(kx - \omega t). \qquad (4)$$

From (2) we see that p' must have the same form as (4) at $z = 0$. The solution which obeys Laplace's equation and (1) and (2) is easily shown to be

$$p' = \bar{\rho} g h \left[\frac{\cosh k(z+H)}{\cosh kH}\right] \cos(kx - \omega t) \qquad (5)$$

Applying (5) in (5) of L5 to find $\partial w'/\partial t$ and then integrating w.r.t. t gives

$$w' = \frac{ghk}{\omega}\left[\frac{\sinh k(z+H)}{\cosh kH}\right] \sin(kx - \omega t) \qquad (6)$$

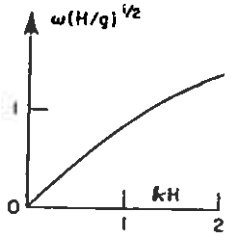
Condition (3) must also be obeyed. Applying (3) to (6) [using (4)] we find:

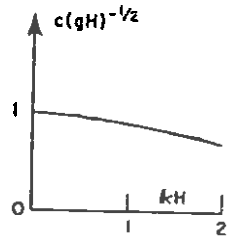
$$w'(0) = \frac{ghk}{\omega}\left[\frac{\sinh kH}{\cosh kH}\right] \sin(kx - \omega t) = \omega h \sin(kx - \omega t)$$

Hence, finally, our dispersion relation:

$$\omega^2 = gk \tanh kH \qquad (7)$$

Scaled frequency (ω) and phase speed ($c = \omega/k$) are plotted below as functions of kH.





We can find u' by applying (5) in (5) of L5 to find $\partial u'/\partial t$ and integrating w.r.t. t:

$$u' = \frac{ghk}{\omega}\left[\frac{\cosh k(z+H)}{\cosh kH}\right] \cos(kx - \omega t) \qquad (8)$$

Instead of continuing with the general case we consider the limits $kH \gg 1$, $kH \ll 1$.

Deep water limit: $kH >> 1$

In this case the dispersion relation reduces to $\omega^2 = gk$ (9)

The phase speed $c = \omega/k$ is thus given by $c^2 = g/k$; (10)

so long waves move faster than short waves (for which there is maritime evidence).

From (9) we find $2\omega\dfrac{d\omega}{dk} = g \Rightarrow c_g = c/2$; (11)

the group velocity of deep water waves is half their phase speed. [It is *not* generally true, for all systems, that the group velocity is less than the phase velocity.]

Eqs (5), (6) and (8) simplify in the deep water limit (since $\tanh kH \to 1$); the velocity components, in particular, are given by:

$$u' = \frac{ghk}{\omega}\exp(kz)\cos(kx-\omega t) \quad ; \quad w' = \frac{ghk}{\omega}\exp(kz)\sin(kx-\omega t) \; . \qquad (12)$$

Everything decays away exponentially below the surface, the decay space-scale being k^{-1} (where k, of course, is the horizontal wavenumber). Eqs (12) can be used to find the trajectories (paths, orbits) of parcels of fluid [the *Lagrangian* view, rather than the *Eulerian* view in terms of velocities]:

$$x - x_0 \approx -\frac{ghk}{\omega^2}\exp(kz_0)\sin(kx_0-\omega t) \; ; \quad z - z_0 \approx \frac{ghk}{\omega^2}\exp(kz_0)\cos(kx_0-\omega t) \qquad (13)$$

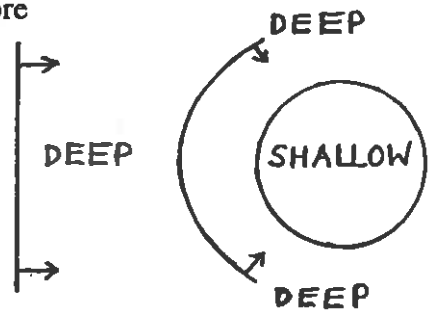
[x_0, z_0 give the mean position of a chosen parcel.] From (13) it is easily seen that the parcel paths are *circles*, whose radii decrease exponentially with depth.

Shallow water limit: $kH << 1$

In this case the dispersion relation reduces to $\omega^2 = gHk^2$, (9)

so the phase speed $c = \omega/k$ is simply $c = \pm\sqrt{gH}$. (10)

Shallow water waves are *non-dispersive*, and their phase speed varies as the square root of the depth H. This accounts in a qualitative way for the fact that sea waves break at large amplitude, and (rather more convincingly) for the occurrence of particularly rough seas in regions of shallow water; refraction of wave-fronts tends to concentrate wave energy where H is small.



The forms of u', w', p' are readily found from (5), (6) and (8) if it is noted that $|z| < H$ as well as $kH << 1$:

$$p' = \overline{\rho}gh\cos(kx-\omega t) \; ; \quad u' = \frac{ghk}{\omega}\cos(kx-\omega t) \; ; \quad w' = \frac{ghk^2}{\omega}(z+H)\sin(kx-\omega t) \; .$$

Thus the pressure perturbation and horizontal velocity are *independent of depth*, and the vertical velocity increases linearly with height above the lower boundary, but is at all levels of order (at most) kHu', and hence is formally negligible. In shallow water waves, columns of water move horizontally to and fro. [It can be shown that the parcel paths for general depth H are ellipses; these degenerate to circles at the deep water limit, and to lines at the shallow water limit.]

Exercise: Starting from (3) *and* (4) *of L5, and assuming that* **u** *and* p' *are independent of height, derive the nonlinear shallow water equations*

$$\frac{D\mathbf{v}}{Dt} + g\nabla_h\eta = 0 \; ; \quad \frac{D\eta}{Dt} + (H+\eta)\nabla_h\cdot\mathbf{v} = 0 \; . \quad [\mathbf{v} = (u, v); \; \nabla_h = (\partial/\partial x \, , \partial/\partial y)]$$

7. Pure acoustic waves; shallow water wave isomorphism; general solution

The pure acoustic wave problem is much easier to set up than the surface water wave problem (in which boundary conditions are crucial). We use the equations of motion for a perfect gas (L4), and linearise about a state of rest. It is convenient to use yet another form of the thermodynamic equation (T1); by using $p = \rho RT$ in (1) of L5 one readily obtains (assuming $Q = 0$)

$$\frac{1}{\gamma p}\frac{Dp}{Dt} - \frac{1}{\rho}\frac{D\rho}{Dt} = 0 \qquad (1)$$

in which $\gamma = c_p/c_v$ is the ratio of the principal specific heats. [Eq(1) is easy to remember if you note that $p\rho^{-\gamma}$ is constant in an adiabatic change.] Linearising (1) about a state of rest gives

$$\frac{1}{\gamma\bar{p}}\frac{\partial p'}{\partial t} - \frac{1}{\bar{\rho}}\frac{\partial \rho'}{\partial t} = 0 \qquad (2)$$

Here, $\bar{p}$ and $\bar{\rho}$ are the basic state pressure and density field upon which the perturbations p' and ρ' are applied. We examine a basic state which is isothermal (temperature $\bar{T}$), and ignore gravity; $\bar{p}$ and $\bar{\rho}$ are constants. Consider motion only in the x direction. [This is not a restrictive choice, since the acoustic waves that we find are longitudinal, and the absence of gravity means that the x axis can point in any direction we like.] The perturbed x momentum and continuity equations are then

$$\frac{\partial u'}{\partial t} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} = 0 \qquad (3)\ ; \qquad \frac{\partial \rho'}{\partial t} + \bar{\rho}\frac{\partial u'}{\partial x} = 0 \qquad (4)$$

Using (2), (4) can be written

$$\frac{\partial p'}{\partial t} + \gamma\bar{p}\frac{\partial u'}{\partial x'} = 0 \qquad (5)$$

Then, from (3) and (5):

$$\frac{\partial^2 p'}{\partial t^2} = \frac{\gamma\bar{p}}{\bar{\rho}}\frac{\partial^2 p'}{\partial x^2} = \gamma R\bar{T}\frac{\partial^2 p'}{\partial x^2} \qquad (6)$$

Equation (6) is the classical wave equation, with general solution

$$p'(x,t) = F(x-ct) + G(x+ct) \qquad (7)$$

where $c = \sqrt{\gamma R\bar{T}}$ is the adiabatic sound speed, and F and G are arbitrary twice-differentiable functions. If the basic state temperature varies slowly, then refraction occurs in essentially the same way as in the shallow water equations (L6) – although in that case, of course, the phase speed varies as the square root of the water depth. Acoustic waves are longitudinal; the velocities and hence displacements are solely parallel to the direction of propagation (x).

Exercise: What happens if a term $\nu\,\partial^2 u'/\partial x^2$ is included on the r.h.s. of (3)?

Shallow water isomorphism

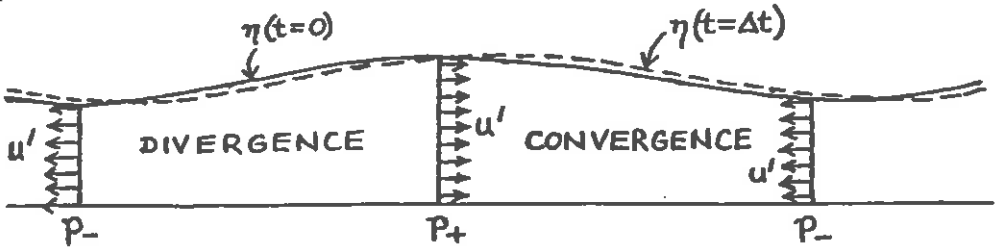
If we linearise the shallow water equations, or take the long wave limit of the more general linearised, incompressible fluid equations studied earlier in L6, we obtain

$$\frac{\partial u'}{\partial t} + g\frac{\partial \eta}{\partial x} = 0 \qquad (8); \qquad \frac{\partial \eta}{\partial t} + H\frac{\partial u'}{\partial x} = 0\ . \qquad (9)$$

The formal similarity of (8) and (9) to (3) and (5) is notable, though in one case we are dealing with long, hydrostatic surface waves on an incompressible liquid, while in

the other we are dealing with acoustic waves in a compressible fluid (perfect gas) without gravity. We have the correspondences $\bar{\rho} g \eta \Leftrightarrow p'$ and $\bar{\rho} g H \Leftrightarrow \gamma \bar{p}$. Shallow water waves and pure acoustic waves can be understood physically and analysed mathematically using the same ideas and methods.

The physical mechanism of shallow water waves and acoustic waves
Although arbitrary (smooth) functions F and G (see(7)) figure in the general solution, it is helpful to particularise to the case of sinusoidal waves. Consider a long surface water wave having the form $\eta = h\cos(kx - \omega t)$. Then [see L6] the velocity component in the x direction has similar form. Consider the case shown:



Convergence to the right of the crest of surface displacement, and divergence to the right of the trough of displacement, will lead to the movement of the pattern of surface displacement to the right. At the same time, the pressure gradient force (associated hydrostatically with the surface displacement) moves the pattern of velocities to the right. Had the pattern of velocities been reversed at the initial time, then the pattern would have moved to the left. The same reasoning may be applied to an acoustic wave. [*Exercise: Carry this through.*] These physical aspects tend to be obscured if we focus attention only on the wave equation: the mathematical treatment delivers a solution, but we need to unwrap it.

General solution, given initial fields of pressure and velocity perturbation
Our discussion of the mechanism of propagation indicates that the pattern of initial velocities is as important as the pattern of initial surface displacements (or pressures). We can use (8) and (9) [or (3) and (5)] to make this indication mathematically precise and to derive solutions for general (smooth) initial conditions. From (8) and (9) we can derive the classical wave equation and deduce that the general solution (with $c^2 = gH$) is

$$\eta(x,t) = F(x-ct) + G(x+ct) \qquad (10)$$

Suppose that $\eta(x,0)$ is given. Then

$$\eta(x,0) \equiv \eta_0(x) = F(x) + G(x) \qquad (11)$$

If $u'(x,0)$ [the perturbation velocity at $t = 0$] is also given, then, from (8):

$$u'(x,0) \equiv u_0(x) = \frac{g}{c}[F(x) - G(x)] \qquad (12)$$

Hence, we can find F and G from (11) and (12), and then write the general solution as

$$2\eta(x,t) = \eta_0(x-ct) + \eta_0(x+ct) + \frac{c}{g}\left(u_0(x-ct) - u_0(x+ct)\right) \qquad (13)$$

$$2u(x,t) = u_0(x-ct) + u_0(x+ct) + \frac{g}{c}\left(\eta_0(x-ct) - \eta_0(x+ct)\right) \qquad (14)$$

Exercises: Relate (13) *and* (14) *to d'Alembert's solution of the wave equation [L2]. What do the solutions look like in the cases (i)* $u_0 = 0$, $\eta_0 \neq 0$ *(ii)* $u_0 \neq 0$, $\eta_0 = 0$ *?*

8. Acoustic waves in the presence of gravity; Lamb waves

The general problem of wave motion in a compressible atmosphere under gravity is fairly complicated even when the basic state is isothermal and at rest; we postpone treatment of this case until a later lecture. Here we consider the special and important cases of vertically propagating acoustic waves and waves in which the motion is purely horizontal. We establish various properties of an isothermal basic state first.

Variation of pressure, density and potential temperature in an isothermal atmosphere
From the hydrostatic relation it follows that the pressure and density in an isothermal atmosphere at rest vary with height z as

$$\bar{p}(z) = \bar{p}(0)\exp\{-z/\bar{H}\} \qquad (1)$$

and

$$\bar{\rho}(z) = \bar{\rho}(0)\exp\{-z/\bar{H}\}. \qquad (2)$$

$\bar{H} \equiv R\bar{T}/g$ is called the *scale height.*

From the definition of potential temperature we find $\frac{N^2}{g} \equiv \frac{1}{\bar{\theta}}\frac{d\bar{\theta}}{dz} = \frac{1}{\gamma\bar{p}}\frac{d\bar{p}}{dz} - \frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dz}$.

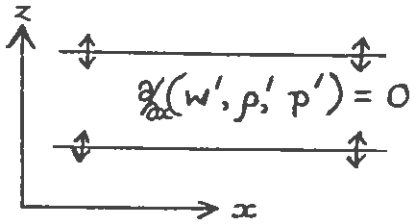
Use of the hydrostatic relation and (2) gives $\frac{N^2}{g} = \frac{1}{\bar{H}} - \frac{g}{c_s^2}$, (3)

in which $c_s = \sqrt{\gamma R\bar{T}}$ is the adiabatic sound speed. Using $c_p - c_v = R$, (3) can then be written in terms of $\bar{T}$ as $N^2 = g^2/c_p\bar{T}$. (4)

Hence $\bar{H}, c_s$ and N are each independent of height in an isothermal atmosphere.

Potential temperature increases exponentially with height: $\bar{\theta} = \bar{\theta}(0)\exp\{gz/c_p\bar{T}\}$.

Vertically propagating acoustic waves
Suppose that all perturbations vary only with height, and that the velocity has no horizontal components. The linearized momentum, continuity and thermodynamic equations about an isothermal rest state are:



$$\frac{\partial w'}{\partial t} + g\frac{\rho'}{\bar{\rho}} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} = 0 \qquad (5)$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial z}(\bar{\rho}w') = 0 \qquad (6)$$

$$\frac{1}{\gamma\bar{p}}\frac{\partial p'}{\partial t} - \frac{1}{\bar{\rho}}\frac{\partial \rho'}{\partial t} + \frac{N^2}{g}w' = 0 \qquad (7).$$

For convenience we drop the primes from now on; thus u, w, p and ρ are perturbation quantities. We also introduce a tracer parameter, n, to multiply the local time derivative term in (5); this parameter takes the value 1 when the associated term is included, and 0 if we choose to omit it. With these changes and additions, and some rearrangement, (5) – (7) become:

$$n\frac{\partial(\bar{\rho}w)}{\partial t} + g\rho + \frac{\partial p}{\partial z} = 0 \qquad (8)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\bar{\rho}w) = 0 \qquad (9);$$

$$\frac{1}{c_s^2}\frac{\partial p}{\partial t} - \frac{\partial \rho}{\partial t} + \frac{N^2}{g}\bar{\rho}w = 0 \qquad (10)$$

From $\partial/\partial t$ (8) into (9): $$g\frac{\partial}{\partial z}(\overline{\rho}w) - \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial z}\right) = n\frac{\partial^2}{\partial t^2}(\overline{\rho}w) \qquad (11)$$

From $\partial/\partial z$ of [(10) + (9)]: $$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t}\right) + \frac{c_s^2 N^2}{g}\frac{\partial}{\partial z}(\overline{\rho}w) = -c_s^2\frac{\partial^2}{\partial z^2}(\overline{\rho}w) \qquad (12)$$

Add (11) and (12) and re-arrange using (3):

$$\left(n\frac{\partial^2}{\partial t^2} - \frac{c_s^2}{\overline{H}}\frac{\partial}{\partial z} - c_s^2\frac{\partial^2}{\partial z^2}\right)\overline{\rho}w = 0 \qquad (13)$$

By noting (2), and using standard methods, the first derivative term in (13) can be removed:

$$\left(n\frac{\partial^2}{\partial t^2} - c_s^2\frac{\partial^2}{\partial z^2} + \frac{c_s^2}{4\overline{H}^2}\right)\sqrt{\overline{\rho}}w = 0 \qquad (14)$$

• Consider the case $n = 1$. [This equation appeared as an example *circa* L4.]

Substituting $\sqrt{\overline{\rho}}w = \exp[i(mz - \omega t)]$ we find $\omega^2 = c_s^2\left(m^2 + \frac{1}{4\overline{H}^2}\right)$. (15)

These acoustic waves modified by gravity are *slightly* supersonic and *slightly* dispersive; however, the vertical wavelength has to be comparable with the scale height before these effects are noticeable.

Exercise: Find and interpet the forms of w, ρ and p in these waves.

• Consider the case $n = 0$. Now the time-dependent term drops out of (14): there are no wave-like solutions. Making the hydrostatic approximation in (4) removes vertically propagating acoustic waves. This is an example of a more general result (of which more later).

Lamb waves

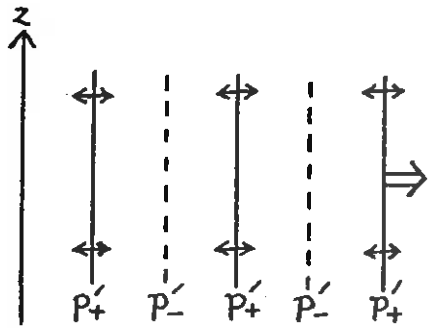
Consider the analysis given in L7 for pure acoustic waves. As we noted in L7, in the absence of gravity it does not matter which way the x axis points. But suppose that gravity *is* present and that the x axis *is* in the horizontal plane (i.e. in the plane perpendicular to the direction of gravity). Then the analysis of L7 carries through in the case of an isothermal basic state; velocity perturbations are purely horizontal so long as the density and pressure perturbations are in hydrostatic balance:

$$\rho' g + \frac{\partial p'}{\partial z} = 0. \qquad (16)$$

Suppose that $p' = P(z)\exp[ik(x \pm c_s t]$, where $P(z)$ is to be determined. From (10) and (16), noting that $w' = 0$, it follows that

$$\frac{g}{c_s^2}P = -\frac{dP}{dz} \Rightarrow P = \exp[-gz/c_s^2]$$

i.e. $p' = \exp[-gz/c_s^2]\exp[ik(x \pm c_s t]$ (17).



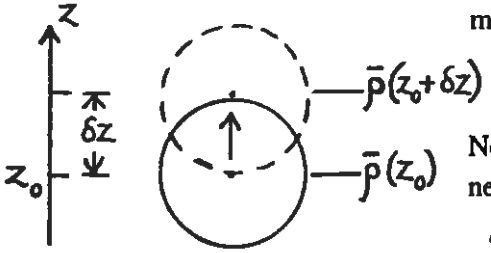
These horizontally propagating acoustic waves are called Lamb waves. They are "edge waves", allowed because the presence of the ground excludes the region $z \to -\infty$. They are *not* removed by the hydrostatic approximation: they have zero vertical velocity.

Exercises: Find ρ′ and u′ corresponding to (17). What happens if the basic state is not isothermal?

9. Internal gravity waves in a stratified incompressible fluid

Waves on the interface between 2 immiscible homogeneous liquids having densities $\overline{\rho}$ and $\overline{\rho}+\Delta\overline{\rho}$ are similar to surface waves. The main difference is that g is replaced by *reduced gravity* $g\Delta\overline{\rho}/\overline{\rho}$ (so that interfacial waves move more slowly than surface waves). ***Internal gravity waves*** can occur in fluids which are *stratified.* A stratified incompressible fluid has a continuous variation of mean density with height; a stratified perfect gas has a continuous variation of potential temperature with height. This lecture deals with the incompressible fluid case.

Parcel oscillations: the buoyancy or Brunt-Väisällä frequency
Consider an incompressible fluid, at rest, in which the density decreases with height. [This decrease could be due to an increase of temperature with height.] Suppose that a small parcel of fluid at height z_0 is displaced upwards a small distance δz. According to the definition of an incompressible fluid, each parcel retains its density as it moves: $D\rho/Dt = 0$. Hence the displaced parcel will obey the momentum equation



$$\frac{Dw}{Dt} = -g - \frac{1}{\overline{\rho}(z_0)}\left.\frac{\partial p}{\partial z}\right|_{z_0+\delta z} \qquad (1)$$

Now $p = \overline{p}(z) + p'(x,y,z,t)$; but let us assume that p' makes a negligible contribution to the pressure gradient term in (1). Then

$$\left.\frac{\partial p}{\partial z}\right|_{z_0+\delta z} \cong \left.\frac{\partial \overline{p}}{\partial z}\right|_{z_0+\delta z} = -g\overline{\rho}(z_0+\delta z) \cong -g\left[\overline{\rho}(z_0) + \delta z\left.\frac{d\overline{\rho}}{dz}\right|_{z_0}\right],$$

and (1) becomes, for small displacements δz:

$$\frac{D^2}{Dt^2}(\delta z) = \frac{Dw}{Dt} = \frac{g\delta z}{\overline{\rho}(z_0)}\left.\frac{d\overline{\rho}}{dz}\right|_{z_0} = -N^2\delta z \qquad (2)$$

The quantity $N = \sqrt{\dfrac{-g}{\overline{\rho}}\dfrac{d\overline{\rho}}{dz}}$ is called the buoyancy frequency, or Brunt-Väisällä frequency. It is the oscillation frequency of a parcel of fluid displaced vertically from its equilibrium position. When $N^2 < 0$ (i.e. $d\overline{\rho}/dz > 0$), (2) gives an imaginary frequency implying exponential growth of the displacement δz; as expected, this top-heavy configuration is unstable.
Exercise: [Important] *Suppose that the parcel of fluid is displaced, but is constrained to move along a line inclined at an angle θ to the vertical. Show that the frequency of oscillation is N*cosθ.

Expansion of the quantity $g + (1/\rho)\partial p/dz$; "buoyancy"
Suppose the basic state is a state of rest in which $\overline{\rho} = \overline{\rho}(z)$ *and* $\overline{p} = \overline{p}(z)$; then $\overline{\rho}g + \partial\overline{p}/\partial z = 0$.
Let us work carefully through the expansion of $g + (1/\rho)\,\partial p/dz$, *where* $\rho = \overline{\rho} + \rho'(x,y,z,t)$, $p = \overline{p}(z) + p'(x,y,z,t,)$ and $|\rho'|/\overline{\rho}, |p'|/\overline{p} << 1$:

$$g + \frac{1}{\rho}\frac{\partial p}{\partial z} = g + \frac{1}{(\overline{\rho}+\rho')}\frac{\partial}{\partial z}(\overline{p}+p') = g + \frac{1}{\overline{\rho}}\left(1+\frac{\rho'}{\overline{\rho}}\right)^{-1}\left(\frac{\partial p'}{\partial z} - \overline{\rho}g\right)$$

Gravity –PGF

$$\cong \cancel{g} - \cancel{g} + \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} + g\frac{\rho'}{\overline{\rho}}$$

Clearly, the basic state part of $(1/\rho)\partial p/\partial z$ has cancelled with g, leaving 2 terms (after second and higher order terms have been neglected). The first of these terms evidently derives from the perturbation pressure gradient. The second term, $g\rho'/\overline{\rho}$, has come from the $1/\rho$ part of the pressure gradient force term (acting in concert with the basic state pressure gradient). It represents the effect of the density

perturbation ρ' on $(1/\rho)\partial p/\partial z$ in a linearised approximation. So, for example, when the density perturbation is positive, the PGF component per unit mass is numerically smaller than it would otherwise be. The term $g\rho'/\overline{\rho}$ is often called the *buoyancy term*; our discussion makes clear that its origin is the PGF term. We shall follow common practice, though reluctantly. [Lighthill's textbook has a 12 page index which does not contain the word buoyancy.]

Our discussion also shows that pressure perturbations should be considered as well as density perturbations, whereas our parcel argument was a little vague in this respect. We now carry out a full perturbation analysis of the equations to see what the possibilities for wave motion are.

Linearised analysis, dispersion relation and interpretation

The relevant nonlinear equations are:

$$\frac{D\mathbf{u}}{Dt} + g\frac{\rho'}{\overline{\rho}}\mathbf{k} + \frac{1}{\overline{\rho}}\nabla p' = 0; \qquad \frac{D}{Dt}(\overline{\rho}+\rho') = 0; \qquad \nabla\cdot\mathbf{u} = 0$$

Linearise about the rest state and consider only perturbations independent of y:

$$\frac{\partial u'}{\partial t} + \frac{\partial}{\partial x}\left(\frac{p'}{\overline{\rho}}\right) = 0 \quad (3); \qquad \frac{\partial w'}{\partial t} + g\frac{\rho'}{\overline{\rho}} + \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} = 0 \quad (4);$$

$$g\frac{\partial}{\partial t}\left(\frac{\rho'}{\overline{\rho}}\right) - N^2 w' = 0 \quad (5); \qquad \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \quad (6)$$

Assume for simplicity that N^2 is independent of height, and approximate the third term in (4) by $\partial/\partial z(p'/\overline{\rho})$. Manipulation of (3) - (6) then shows that

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right]w' + N^2\frac{\partial^2 w'}{\partial x^2} = 0 \qquad (7)$$

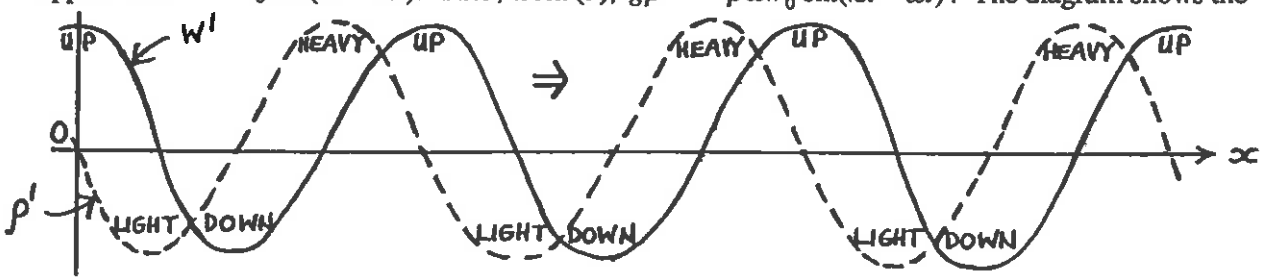
Solutions of the form $w' = \exp\{i(kx + mz - \omega t\}$ must satisfy the dispersion relation

$$\omega^2 = \frac{N^2k^2}{\left(k^2+m^2\right)} \qquad (8)$$

In the case $m = 0$, (8) reduces to $\omega = \pm N$, and (3) - (6) reduce to $u' = p' = 0$ and

$$\frac{\partial w'}{\partial t} + g\frac{\rho'}{\overline{\rho}} = 0 \qquad (9); \qquad g\frac{\partial}{\partial t}\left(\frac{\rho'}{\overline{\rho}}\right) - N^2 w' = 0 \qquad (10)$$

Suppose that $w' = w_0\cos(kx - \omega t)$. Then, from (9), $g\rho' = -\overline{\rho}\omega w_0 \sin(kx - \omega t)$. The diagram shows the



situation at $t = 0$, assuming $\omega = +N$. The phase relationship between w' and ρ' ensures that ρ' moves the w' pattern to the right (from (9)), and that w' moves the ρ' pattern to the right (from (10)). Thus both patterns, and hence the wave itself, translate to the right.

Exercises:

1. Show by a similar analysis that the $m = 0$ wave moves to the left if $\omega = -N$.

2. From the continuity equation (6) show that parcel displacements are perpendicular to the wave vector $\mathbf{k} = (k,m)$ even when $m \neq 0$. Hence show that the dispersion relation (8) is consistent with the result that the frequency of oscillation of parcels constrained to move at an angle θ to the vertical is $\omega = \pm N\cos\theta$.

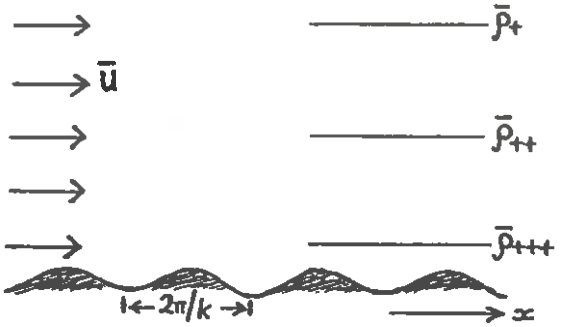
3. Show that the group velocity $\mathbf{c}_g = (\partial\omega/\partial k, \partial\omega/\partial m)$ is perpendicular to the wave vector $\mathbf{k}$.

10. Stratified incompressible flow over wavy terrain

In L9 we saw that "buoyancy" oscillations and "internal gravity" waves in a stratified incompressible fluid have frequencies $|\omega|$ less than or equal to N. If $|\omega| = N$, particle motions are purely vertical; if $|\omega| < N$, they are inclined to the vertical at an angle θ, where $\cos\theta = |\omega|/N$. There are many interesting and important consequences. In the laboratory, a stratified (nearly incompressible) fluid can be subjected to local mechanical forcing at a fixed frequency ω_F; the response is spatially confined if $\omega_F > N$, but domain-filling if $\omega_F < N$.

Steady stratified flow over sinusoidal orography

A more relevant meteorological context is flow over hills and mountains. In uniform flow $\overline{U}$ over sinusoidal corrugations of a rigid lower boundary $h = h_0 \sin kx$, parcels of (incompressible, stratified) fluid experience forcing at frequency $\overline{U}k$; the response is confined close to the boundary if $\overline{U}k > N$, but is domain-filling if $\overline{U}k < N$. Since $\overline{U}$ is uniform, the steady-state linearised equations are the same as (3)-(6) of L9, but with $\partial/\partial t$ replaced by $\overline{U}\,\partial/\partial x$:



$$\overline{U}\frac{\partial u'}{\partial x} + \frac{\partial}{\partial x}\left(\frac{p'}{\overline{\rho}}\right) = 0 \qquad (1); \qquad \overline{U}\frac{\partial w'}{\partial x} + g\frac{\rho'}{\overline{\rho}} + \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} = 0 \qquad (2);$$

$$\overline{U}\frac{\partial}{\partial x}\left(\frac{\rho'}{\overline{\rho}}\right) - \frac{N^2}{g}w' = 0 \qquad (3); \qquad \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \qquad (4)$$

Either by analogy with (7) of L9, or from (1)-(4) above with the same assumptions as in L9, we obtain

$$\overline{U}^2\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right]w' + N^2 w' = 0 \qquad (5)$$

(upon integrating twice w.r.t. x). At the corrugated boundary $h = h_0 \sin kx$ we require

$$w' = \overline{U}\frac{\partial h}{\partial x} = \overline{U}kh_0 \cos kx \qquad (6)$$

We shall apply (6) at $z = 0$ (the mean height of the boundary), thus assuming h_0 to be small in some sense. Because $\overline{U}$ and N^2 are constants, solutions of our interior equation (5) exist in the form

$$w' = w_0 \exp\{i(kx + mz)\} \qquad (7)$$

with (by substitution in (5))

$$\overline{U}^2(k^2 + m^2) = N^2 \qquad \Rightarrow \qquad m^2 = \frac{N^2}{\overline{U}^2} - k^2 \equiv \overline{m}^2 \qquad (8)$$

Everything depends on the sign of $\overline{m}^2$ defined by (8).

i) $\overline{m}^2 > 0$ **i.e.** $N^2 > k^2\overline{U}^2$

In this "propagating" case, the frequency of forcing (as felt by a travelling parcel of fluid) is less than the mean state buoyancy frequency, so there is a free oscillation

about the basic state which can match it. This oscillation can extend throughout the region $z > 0$. Solutions which obey (5) and (6) are

$$w' = \bar{U}kh_0 \cos(kx \pm \bar{m}z) \qquad (9)$$

Notice the sign ambiguity in (9); $\cos(kx + \bar{m}z)$ has a very different spatial variation from that of $\cos(kx - \bar{m}z)$. We postpone choice between these two possibilities until after (ii). In either case, parcel paths relative to the mean flow are straight lines.

(ii) $\bar{m}^2 < 0$ **i.e.** $N^2 < k^2\bar{U}^2$

The solutions are now $w' = \bar{U}kh_0 \exp(\pm\bar{\mu}z)\cos kx$ where $\bar{\mu}^2 = -\bar{m}^2 = k^2 - \dfrac{N^2}{\bar{U}^2}$. In this case it is obvious which is the silly solution; we reject the one that grows exponentially with z, and so obtain

$$w' = \bar{U}kh_0 \exp(-\bar{\mu}z)\cos kx \qquad (10)$$

In this "trapped" case the frequency of forcing is greater than the mean state buoyancy frequency, so there is no free oscillation to match it, and the response is confined to the vicinity of the boundary. From (10) and (1) – (4), u', p' and ρ' may be calculated: each varies with x as $\pm\sin kx$. The pressure perturbations are thus in phase with $h = h_0 \sin kx$, and so there is no net horizontal force on the lower boundary.

Exercise: Show that the parcel paths, relative to the mean flow $\bar{U}$, are circles of exponentially decreasing radius with mean distance above $z = 0$.

Choosing the correct solution in the "propagating" case

From (1) and (4) we can find the pressure perturbation implied by (9):

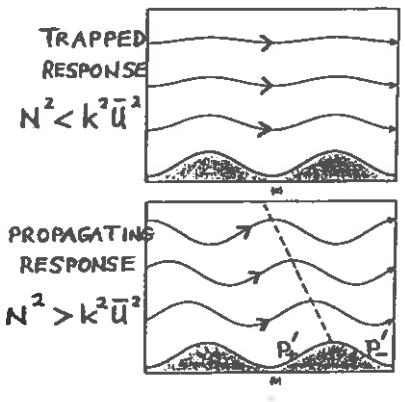
$$\frac{\partial u'}{\partial x} = -\frac{\partial w'}{\partial z} = \pm\bar{U}kh_0\bar{m}\sin(kx \pm \bar{m}z) \quad \Rightarrow \quad u' = \mp\bar{U}h_0\bar{m}\cos(kx \pm mz)$$

$$\Rightarrow \quad p' = \pm\bar{\rho}\bar{U}^2h_0\bar{m}\cos(kx \pm \bar{m}z) \qquad (11)$$

In *both cases*, the pressure perturbation at $z = 0$ is out of phase with $h = h_0 \sin kx$; we choose the sign in (11) so that the net horizontal force *on* the lower boundary (in the direction of $\bar{U}$) is positive. This net force *on* the lower boundary, per wavelength, per unit width of the flow, is

$$\mathcal{S} = \int_0^{2\pi/k} (p'(0)\,\partial h/\partial x)dx = \pm\int_0^{2\pi/k} \bar{\rho}\bar{U}^2k\bar{m}h_0^2\cos^2 kxdx$$

$$\therefore \mathcal{S} = \pm\pi\bar{\rho}\bar{U}^2\bar{m}h_0^2 \qquad (12)$$



We require the + sign in (12), which directs our choice of the + (upper) sign in (9):

$$w' = \bar{U}kh_0 \cos(kx + \bar{m}z) \qquad (13)$$

Hence (11) becomes

$$p' = \bar{\rho}\bar{U}^2h_0\bar{m}\cos(kx + \bar{m}z) \qquad (14)$$

Exercises: 1. Using the dispersion relation (8) of L9, examine the group velocity properties of $\exp\{i(kx \pm mz - \omega t)\}$ waves, and relate these to the selection of the $\cos(kx + \bar{m}z)$ solution in the steady, forced wave problem.

2. Find the parcel paths relative to the lower boundary in the trapped (i) and propagating (ii) cases. The answers are drawn in the diagrams above.

11. More on flow over wavy terrain; buoyancy oscillations in a stratified, perfect gas atmosphere

The problem we considered in L10 was very simple, but it has several interesting implications, which we examine in this lecture. For applications to the atmosphere, one of the least satisfactory aspects of the model used in L9 and L10 is its restriction to incompressible fluids. We also examine in this lecture the form of the buoyancy frequency N for a perfect gas atmosphere; this quantity will figure prominently in our study in L12 of the wave motions possible in a stratified, compressible fluid.

Wavenumber dependence of the response to wavy forcing

Our emphasis in L10 was on the occurrence of trapped or propagating wave responses and the ability of parcel oscillation theory to explain the transition between the two types (at $N = k\overline{U}$). We considered only a single wavenumber of forcing, but the linearity of the problem means that we can superpose individual solutions to obtain the response to forcing which is distributed over many wavenumbers. For example, if the orography is odd about $x = 0$ and is represented as a Fourier sine integral

$$h(x) = \int_0^\infty \hat{h}(k) \sin kx dk \ , \qquad (1)$$

the solution may be obtained by using the methods of L10 with $h_0 = \hat{h}(k)$, and then integrating the result over k. In technical respects this integration may be a difficult operation, but the physical principle is clear: *Fourier components having* $|k| > N/\overline{U}$ *will give a "trapped" contribution* [response decaying exponentially with height] *and Fourier components having* $|k| < N/\overline{U}$ *will give a "propagating" contribution* [filling the domain $z > 0$ in our problem]. The response seen at large heights will be dominated by the low wavenumber components of the forcing; components having $|k| > N/\overline{U}$ will contribute little, and those having $|k| >> N/\overline{U}$ hardly at all. In a sense, the stratified fluid acts as a low pass filter to the forcing supplied by the terrain.

Generalisations and complications

Our simple model is not only linear, it assumes a uniform basic flow $\overline{U}$ and neglects variations in the y direction. In reality, the flow and the terrain rarely approach these idealisations, and the flow direction as well as its magnitude may vary in space and time. Some of these effects can be included in analytical and theoretical treatments. Interesting effects occur even if $\overline{U}$ varies only with height, since reflection, refraction and transmission phenomena may occur. We haven't the time now to go into these important issues, but will return to some of them towards the end of the course.

Consequences of the hydrostatic approximation

What happens if we replace the vertical momentum equation (2) of L10 by the hydrostatic version? This can be investigated by writing L10 (2) in the form

$$n\overline{U}\frac{\partial w'}{\partial x} + g\frac{\rho'}{\overline{\rho}} + \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} = 0 \qquad (2)$$

In (2), $n = 1$ gives the full, non-hydrostatic form, and $n = 0$ is the hydrostatic version.

Elimination of u', ρ' and p' from (1), (3), (4) of L10 and (2) leads to

$$\overline{U}^2\left[n\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right]w' + N^2w' = 0 \qquad (3)$$

Substitution of $w' = w_0\exp\{i(kx+mz)\}$ (as in L10) into (3) gives

$$m^2 = \frac{N^2}{\overline{U}^2} - nk^2 \qquad (4)$$

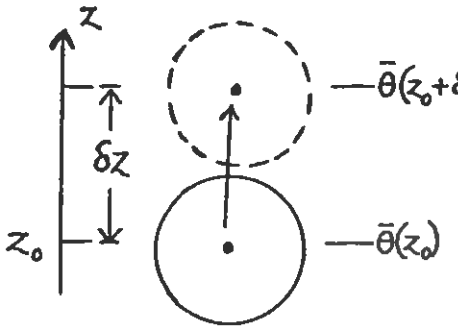
It is clear from (4) that *the hydrostatic approximation* ($n = 0$) *eliminates trapping completely (in this problem); whatever the value of the wavenumber k, the response will be propagating if the hydrostatic approximation is applied.*
[*Exercise:* Do you find this result surprising? Consider what happens to buoyancy oscillations if the hydrostatic approximation is made.]

Validity of the hydrostatic approximation
Under what circumstances is the hydrostatic version ($n = 0$) of (4) a good approximation? We require $k^2\overline{U}^2 << N^2$. In terms of the horizontal wavelength $\lambda = 2\pi/k$ this becomes $\lambda >> 2\pi\overline{U}/N$. Putting $\overline{U}$ = 10ms^{-1}, and $N = 10^{-2}$s^{-1}, gives $\lambda >>$ 6km. Anyone running or building a numerical model intended to resolve motion having a wavelength of 20 km or less should make sure that non-hydrostatic effects are represented.

Buoyancy frequency for a stratified perfect gas atmosphere
The value of N quoted in the previous paragraph anticipated a treatment of buoyancy oscillations in a compressible atmosphere. Consider the displaced parcel of L9; if the fluid is compressible, the parcel's density will vary with the ambient pressure, and so it will not remain constant as the parcel moves in the vertical. Retaining other assumptions as before (L9), the momentum equation describing the motion of the parcel may be written as:

$$\frac{Dw}{Dt} + g + \frac{1}{\rho}\frac{\partial\overline{p}}{\partial z} = 0 \quad\Rightarrow\quad \frac{Dw}{Dt} + g\left(\frac{\rho-\overline{\rho}}{\rho}\right) = 0 \qquad (5)$$

Here w and ρ refer to the parcel (at $z = z_0 + \delta z$) and the basic state is in hydrostatic balance. Since, by assumption, the parcel experiences the pressure of its surroundings, successive application of the perfect gas law ($p = \rho RT$) and the definition of potential temperature [$\theta = T\left(p_{ref}/p\right)^{R/c_p}$] shows that



$$\left(\frac{\rho-\overline{\rho}}{\rho}\right) = \left(\frac{\overline{T}-T}{\overline{T}}\right) = \left(\frac{\overline{\theta}-\theta}{\overline{\theta}}\right) \qquad (6)$$

Now $\theta = \theta(z_0+\delta z) = \overline{\theta}(z_0)$ (because the parcel retains its potential temperature as it moves).

Thus $\left(\frac{\overline{\theta}-\theta}{\overline{\theta}}\right) \cong \frac{1}{\overline{\theta}}\frac{d\overline{\theta}}{dz}\delta z$ and so (5) becomes

$$\frac{D^2}{Dt^2}\delta z + \frac{g}{\overline{\theta}}\frac{d\overline{\theta}}{dz}\delta z = 0$$

Thus the buoyancy frequency in a stratified perfect gas atmosphere is $N = \sqrt{\frac{g}{\overline{\theta}}\frac{d\overline{\theta}}{dz}}$.

Oscillations at an angle θ to the vertical have a frequency $N\cos\theta$, since the argument giving the corresponding result for an incompressible fluid is purely geometric.

12. Waves in an isothermal, perfect gas atmosphere: the perturbation equations

In L9-11 we studied plane waves propagating in an arbitrary direction in a stratified incompressible fluid. In L8 we studied purely vertically propagating waves in an isothermal, perfect gas atmosphere, and also a special sort of horizontally propagating wave (the Lamb wave) in which the vertical velocity is everywhere zero. In this lecture we extend the treatment of L8 to waves propagating in an arbitrary direction, without restriction on the vertical velocity or indeed any other field. We are interested in the consequences of omitting certain terms from the equations, and so will introduce tracer/switch parameters early on. As usual, the basic state is isothermal and at rest, and - for algebraic simplicity – we consider only motion independent of one horizontal direction (y). Also as usual, an overbar denotes the basic state and a prime the perturbation: e.g. $p = \overline{p}(z) + p'(x,z,t)$.

Linearised x-momentum, continuity and thermodynamic equations
These are fairly straightforward:

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial u'}{\partial t} + \frac{\partial}{\partial x}\left(\frac{p'}{\overline{\rho}}\right) = 0 \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{\rho'}{\overline{\rho}}\right) + \left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\right) + \frac{w'}{\overline{\rho}}\frac{\partial \overline{\rho}}{\partial z} = 0 \tag{2}$$

$$\frac{D\theta}{Dt} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{\theta'}{\overline{\theta}}\right) + \frac{N^2}{g}w' = 0 \tag{3}$$

Linearised equation of state
Eqs (1) – (3) have p', ρ' and θ' as their thermodynamic variables. From $p = \rho RT$ and $\theta = T\left(p_{ref}/p\right)^{R/c_p}$ [L5, (1C)] we have $\rho R\theta = p\left(p_{ref}/p\right)^{R/c_p}$. Taking logs and linearising gives $\theta'/\overline{\theta} = p'/\overline{p} - \rho'/\overline{\rho}$, or, in terms of $c_s = \sqrt{\gamma R\overline{T}}$:

$$\frac{\theta'}{\overline{\theta}} = \frac{1}{c_s^2}\frac{p'}{\overline{\rho}} - \frac{\rho'}{\overline{\rho}} \tag{4}$$

Linearised z-momentum equation

$$\frac{Dw}{Dt} + g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial w'}{\partial t} + g\frac{\rho'}{\overline{\rho}} + \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} = 0 \tag{5}$$

Contemplation of (1) – (4) shows that the quantity $p'/\overline{\rho}$ is emerging from the algebraic haze. We aim to write (5) in terms of $p'/\overline{\rho}$. With

$$\overline{H} = -\left(1/\overline{\rho}\right)d\overline{\rho}/dz \quad (= R\overline{T}/g), \tag{6}$$

(5) can be written

$$\frac{\partial w'}{\partial t} + g\frac{\rho'}{\overline{\rho}} + \left(\frac{\partial}{\partial z} - \frac{1}{\overline{H}}\right)\left(\frac{p'}{\overline{\rho}}\right) = 0.$$

Now apply (4) and the relationship $N^2/g = 1/\overline{H} - g/c_s^2$ [L8, (3)]:

$$\frac{\partial w'}{\partial t} - g\frac{\theta'}{\overline{\theta}} + \left(\frac{\partial}{\partial z} - \frac{N^2}{g}\right)\left(\frac{p'}{\overline{\rho}}\right) = 0 \tag{7}$$

We now introduce tracer parameters and adopt some simplifying notation.

Perturbation equations with tracers and new variables

Define $p'/\overline{\rho} \equiv \pi$, $\rho'/\overline{\rho} \equiv R$, $\theta'/\overline{\theta} \equiv S$ and introduce tracers n_w, n_ρ, n in (1) – (4), (7); also drop the primes on u' and w':

$$\frac{\partial u}{\partial t} + \frac{\partial \pi}{\partial x} = 0 \quad (8) \qquad n_w \frac{\partial w}{\partial t} - gS + \left(\frac{\partial}{\partial z} - n\frac{N^2}{g}\right)\pi = 0 \quad (9)$$

$$g\frac{\partial S}{\partial t} + N^2 w = 0 \quad (10) \qquad n_\rho \frac{\partial R}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - \frac{w}{\overline{H}} = 0 \quad (11)$$

$$S = \frac{\pi}{c_s^2} - R \quad (12)$$

Tracer n_w tracks the $\partial w/\partial t$ term in the vertical momentum equation; $n_w = 0$ when the hydrostatic approximation is made, $n_w = 1$ when it is not. Tracer n_ρ tags $\partial p/\partial t$ - the "dynamic compressibility" term in the continuity equation; $n_\rho = 0$ signifies use of an "incompressible" form. Tracer n follows the term $-N^2\pi/g$ in the vertical momentum equation; it is both necessary and worthwhile to explain its presence.

Filtering acoustic waves; energy consistency

Acoustic waves are a nuisance in numerical models of the atmosphere. They are meteorologically unimportant, but have much higher frequencies than gravity waves, and so consume a great deal of computational effort. A major aim of meteorological modelling over the past 50 years has been to find how to get rid of ("filter") acoustic waves whilst doing the least damage to the description of meteorologically important motion. Some sort of hydrostatic approximation and some sort of incompressibility approximation are the two main candidates - which our tracer scheme in the simple linear equations (8)-(12) is equipped to examine. One indicator of the dependability of a proposed scheme of approximation is the quality of its conservation properties, for example how it treats energy conservation. An energy equation may be formed by multiplying (8) by u, (9) by w, and adding:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\left[u^2 + n_w w^2\right]\right) = gwS + n\frac{N^2}{g}w\pi - u\frac{\partial \pi}{\partial x} - w\frac{\partial \pi}{\partial z} \quad (13)$$

Now write the last two r.h.s. terms as a divergence, and apply (11):

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\left[u^2 + n_w w^2\right]\right) = gwS + n\frac{N^2}{g}w\pi - n_\rho \pi\frac{\partial R}{\partial t} + \frac{\pi w}{\overline{H}} - \frac{\partial}{\partial x}(u\pi) - \frac{\partial}{\partial z}(w\pi) \quad (14)$$

$$\downarrow (10) \qquad \downarrow (12),(10)$$

$$-\frac{g^2}{N^2}S\frac{\partial S}{\partial t} \qquad -n_\rho \pi\left(\frac{1}{c_s^2}\frac{\partial \pi}{\partial t} + \frac{N^2}{g}w\right)$$

Make the indicated replacements, multiply by $\overline{\rho}$ and use the definition of $\overline{H}$ [(6)]:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{\rho}\left[u^2 + n_w w^2 + \frac{g^2}{N^2}S^2 + n_\rho \frac{p^2}{\overline{\rho}^2 c_s^2}\right]\right) = \left(n - n_\rho\right)\frac{N^2}{g}wp - \frac{\partial}{\partial x}(up) - \frac{\partial}{\partial z}(wp) \quad (15)$$

- If $n_w = 0$ (hydrostatic approximation) w does not contribute to the kinetic energy.
- If $n_\rho = 0$ (incompressibility approximation) p does not contribute to the quadratic energy quantity on the l.h.s. of (15).
- If $n = n_\rho = 1$ then the r.h.s. term in $N^2 wp/g$ **vanishes** (see Gill 1982, Eq (6.14.7)); it is therefore entirely unphysical. To make sure it never appears, we require $n = n_\rho$, whether $n_\rho = 0$ or 1. **Hence we set $n = n_\rho$ in (9) to retain energy conservation.**

13. Waves in an isothermal, perfect gas atmosphere: dispersion equation for mixed acoustic/gravity waves; "soundproofing"

In L12 we derived linear perturbation equations and set up a tracer scheme for terms we might wish to omit. In terms of $p'/\overline{\rho} \equiv \pi$, $\rho'/\overline{\rho} \equiv R$, $\theta'/\overline{\theta} \equiv S$ and with tracers n_w, n_ρ these equations are

$$\frac{\partial u}{\partial t} + \frac{\partial \pi}{\partial x} = 0 \quad (1) \qquad n_w \frac{\partial w}{\partial t} - gS + \left(\frac{\partial}{\partial z} - n_\rho \frac{N^2}{g}\right)\pi = 0 \quad (2)$$

$$g\frac{\partial S}{\partial t} + N^2 w = 0 \quad (3) \qquad n_\rho \frac{\partial R}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - \frac{w}{\overline{H}} = 0 \quad (4)$$

$$S = \frac{\pi}{c_s^2} - R \quad (5)$$

The occurrence of n_ρ in (2) as well as (4) ensures consistent energetics (see L12).

We now derive from (1) - (5) a dispersion relation for sinusoidal waves and examine the consequences of omitting the traced terms in (2) and (4). Manipulation of (1) - (5) is eased by the fact that N^2, $\overline{H}$ and c_s^2 are all constants for our isothermal basic state.

Exercise for the curious and/or enthusiastic: Show that (1)-(5) themselves are essentially unchanged if the basic state is not isothermal.

Equations relating w and π (= $p/\overline{\rho}$)

Elimination of S between (2) and (3) gives

$$\left(n_w \frac{\partial^2}{\partial t^2} + N^2\right) w + \left(\frac{\partial}{\partial z} - n_\rho \frac{N^2}{g}\right)\frac{\partial \pi}{\partial t} = 0 \quad . \qquad (6)$$

Another equation involving only w and π may be obtained by eliminating R between (4) and (5), and then using (2) and (1) successively to eliminate S and u from the result:

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z} + n_\rho \frac{N^2}{g} - \frac{1}{\overline{H}}\right) w + \left(\frac{n_\rho}{c_s^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\pi = 0 \quad . \qquad (7)$$

We must eliminate w (or π) between (6) and (7)! But let's do something easier first.

Lamb waves

If $w = 0$ everywhere, then (6) $\Rightarrow$

$$\left(\frac{\partial}{\partial z} - n_\rho \frac{N^2}{g}\right)\frac{\partial \pi}{\partial t} = 0 \qquad (8)$$

and (7) $\Rightarrow$

$$\left(\frac{n_\rho}{c_s^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\pi = 0 \qquad (9)$$

- If $n_\rho = 1$ then (irrespective of the value of n_w) we recover the Lamb wave (of L8).
- If $n_\rho = 0$ then (9) shows that there are no Lamb waves. *Making an incompressibility approximation in the continuity equation removes Lamb waves.*

Equation for w

Having disposed of the case $w = 0$, we can now happily (?) eliminate π between (6) and (7) by cross differentiation:

$$\left(\frac{\partial}{\partial z} + n_\rho \frac{N^2}{g} - \frac{1}{\overline{H}}\right)\left(\frac{\partial}{\partial z} - n_\rho \frac{N^2}{g}\right)\frac{\partial^2 w}{\partial t^2} - \left(n_w \frac{\partial^2}{\partial t^2} + N^2\right)\left(\frac{n_\rho}{c_s^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) w = 0 \quad (10)$$

By gathering together the $\partial^2 w/\partial t^2$ terms in (10), noting that $n_\rho^2 = n_\rho$, and again using the relationship $N^2/g = 1/\overline{H} - g/c_s^2$ [L8, (3); also L12] we finally obtain

$$\left(n_w \frac{\partial^2}{\partial t^2} + N^2\right)\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial^2}{\partial z^2} - \frac{1}{\overline{H}}\frac{\partial}{\partial z} - \frac{n_\rho n_w}{c_s^2}\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 w}{\partial t^2} = 0 \qquad (11)$$

Dispersion relation

Eq (11) is 4th order in time if n_w and n_ρ are both unity, but if either is zero then it is 2nd order in time. Making the hydrostatic or incompressibility approximations thus radically changes the mathematical nature of the forecasting problem. To examine the situation in less general terms, consider the behaviour of wave-like solutions of (11) of the form

$$w \propto \exp\left(z/2\overline{H}\right)\exp\{i(kx + mz - \omega t)\} \qquad (12)$$

(cf. the form used in L8 for vertically propagating acoustic waves). For (12) we have

$$\frac{\partial^2}{\partial t^2} \to -\omega^2;\quad \frac{\partial^2}{\partial x^2} \to -k^2;\quad \frac{\partial}{\partial z} \to im + \frac{1}{2\overline{H}};\quad \frac{\partial^2}{\partial z^2} \to -m^2 + \frac{im}{\overline{H}} + \frac{1}{4\overline{H}^2},$$

and substitution in (11) leads to the *acoustic/gravity wave dispersion equation:*

$$\omega^4 n_\rho n_w - \omega^2 c_s^2\left(n_w k^2 + m^2 + \frac{1}{4\overline{H}^2}\right) + N^2 c_s^2 k^2 = 0 \qquad (13)$$

We have 3 cases to consider (not 4, because if $n_w = 0$ the value of n_ρ is immaterial).

(i) $n_w = n_\rho = 1$; **no terms left out of (1) – (5)**

$$\omega^4 - \omega^2 c_s^2\left(k^2 + m^2 + \frac{1}{4\overline{H}^2}\right) + N^2 c_s^2 k^2 = 0 \qquad (14)$$

Each solution of (14), in terms of k and m, is a dispersion relation. It isn't difficult to write the exact solutions in a fairly compact form, but instead of doing this we shall use some very good approximations. Eq (14) has two pairs of solutions. The high frequency pair gives a dominant balance between the ω^4 and ω^2 terms:

$$\omega^2 \approx \omega_a^2 \equiv c_s^2\left(k^2 + m^2 + 1/4\overline{H}^2\right) \quad \Leftarrow \textbf{ acoustic modes} \qquad (15)$$

The low frequency pair gives a dominant balance between the ω^2 and ω^0 terms:

$$\omega^2 \approx \omega_g^2 \equiv N^2 k^2/\left(k^2 + m^2 + 1/4\overline{H}^2\right) \quad \Leftarrow \textbf{ gravity modes} \qquad (16)$$

Exercise: Show that $|\omega_a|$ (slightly) overestimates the true acoustic mode frequencies, and $|\omega_g|$ (slightly) underestimates the true gravity mode frequencies.

(ii) $n_w = 0$; $n_\rho = 0$ or 1; **hydrostatic approximation in (2)**

$$\omega^2 = N^2 k^2/\left(m^2 + 1/4\overline{H}^2\right) \quad \left(\neq \omega_g^2\right) \qquad (17)$$

- Only gravity modes are left, and their frequencies are changed (cf (16)).

(iii) $n_\rho = 0$; $n_w = 1$; **incompressibility approximation in (4)**

$$\omega^2 = N^2 k^2/\left(k^2 + m^2 + 1/4\overline{H}^2\right) \quad \left(= \omega_g^2\right) \qquad (18)$$

- Only gravity modes are left, and their frequencies are as given by (16).
- This approximation is often called the **anelastic approximation**.
- Both $n_w = 0$ and $n_\rho = 0$ *soundproof* the equations - they remove acoustic waves; but $n_w = 0$ fails to remove Lamb waves, and changes gravity wave phase speeds.

13A Applications and implications of the acoustic/gravity wave dispersion equation; the hydrostatic and anelastic approximations

In L13 we derived the acoustic/gravity wave dispersion equation [(13) of L13] as

$$\omega^4 n_w n_\rho - \omega^2 c_s^2\left(n_w k^2 + m^2 + 1/4\overline{H}^2\right) + N^2 c_s^2 k^2 = 0 \qquad (1)$$

The roots of (1) give the angular frequency ω of small amplitude waves having a vertical velocity field of the form $w \propto \exp\{z/2\overline{H}\}\exp\{i(kx+mz-\omega t)\}$. (2)

n_w and n_ρ are tracer parameters in equations (2) and (4) of L13:

$$n_w \frac{\partial w}{\partial t} - gS + \left(\frac{\partial}{\partial z} - n_\rho \frac{N^2}{g}\right)\pi = 0 \qquad (3)$$

$$n_\rho \frac{\partial R}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - \frac{w}{\overline{H}} = 0 \qquad (4)$$

$n_w = 0$ means the hydrostatic approximation (HA) in (3); $n_\rho = 0$ means the anelastic approximation (AA) in (4). Before reviewing the important implications of (1) for numerical model formulation we use it to obtain the solution of a steady, forced wave problem.

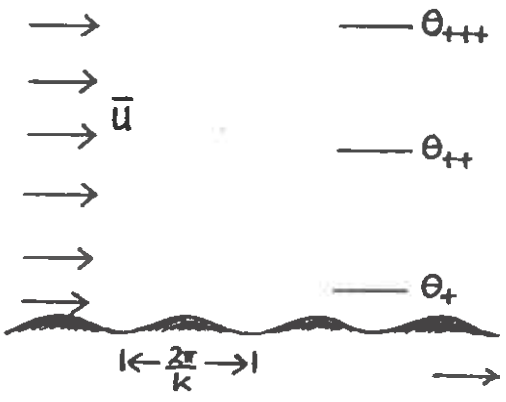
Compressible, isothermal flow over wavy terrain

We analysed stratified incompressible flow over wavy terrain in L10. The perturbation equations (1) – (5) of L13 can be used to repeat that problem for the perfect gas case (with isothermal basic state). But there is an easier way to get the major results. As we noted in L10, the interior equations for the steady wave problem with $\exp(ikx)$ forcing and uniform basic flow $\overline{U}$ are formally the same as those of the free, travelling wave problem (with $\exp[i(kx+mz-\omega t)]$ variation) except that $\overline{U}\partial/\partial x$ replaces $\partial/\partial t$. Thus, to obtain the vertical wavenumber m of the steady wave response to $\exp(ikx)$ forcing, you simply replace ω in (1) by $-\overline{U}k$:

$$\overline{U}^4 k^4 n_w n_\rho - \overline{U}^2 k^2 c_s^2\left(n_w k^2 + m^2 + 1/4\overline{H}^2\right) + N^2 c_s^2 k^2 = 0 \,. \qquad (5)$$

Thus $$m^2 = \frac{N^2}{\overline{U}^2} - n_w k^2\left(1 - n_\rho \frac{\overline{U}^2}{c_s^2}\right) - \frac{1}{4\overline{H}^2} \,. \qquad (6)$$

As argued in L10, the sign of m^2 determines whether the steady wave is trapped ($m^2 < 0$) or propagating ($m^2 > 0$). All the cases of interest can be treated using (6):



- Neither AA nor HA $\left(n_\rho = n_w = 1\right)$

$$m^2 = \frac{N^2}{\overline{U}^2} - k^2\left(1 - \frac{\overline{U}^2}{c_s^2}\right) - \frac{1}{4\overline{H}^2} \qquad (7)$$

- AA but not HA $\left(n_\rho = 0, n_w = 1\right)$

$$m^2 = \frac{N^2}{\overline{U}^2} - k^2 - \frac{1}{4\overline{H}^2} \qquad (8)$$

- HA $\left(n_w = 0, n_\rho = 0 \text{ or } 1\right)$

$$m^2 = \frac{N^2}{\overline{U}^2} - \frac{1}{4\overline{H}^2} \qquad (9)$$

The anelastic approximation (AA) gives an accurate result if $|\overline{U}| \ll c_s$ - a condition that is very well satisfied in the atmosphere! As we found in L11, the hydrostatic approximation (HA) gives an accurate result only if $k^2\overline{U}^2 \ll N^2$, and can give spurious propagation if $k^2\overline{U}^2 > N^2$. Note that (8) and (9) differ from the incompressible fluid results of L10, L11 only as regards the term involving $\left(1/4\overline{H}^2\right)$.

Implications for numerical modelling

If we wish to run a numerical model of the compressible atmosphere it is highly desirable that we use equations that do not support acoustic waves. Unless sophisticated integration schemes are used (and such schemes have their own overheads) the timestep Δt in a numerical model is limited to $\Delta t < d/c$ to ensure stability. Here d is the grid spacing and c is the larger of the flow speed and the largest wave phase speed. If acoustic waves are present, c will invariably be c_s ($\approx 300\ \mathrm{ms}^{-1}$); then $d = 30$ km requires $\Delta t < 100$ s and $d = 300$ m requires $\Delta t < 1$ s. With this background established, we now summarise the properties of our two candidate procedures for removing acoustic waves (as found in L11, L13):

- HA filters acoustic waves except Lamb waves
- AA filters all acoustic waves (including Lamb waves)

We also need to consider how the remaining gravity waves are treated:

- HA eliminates trapping of short steady waves (L11)
- HA changes the phase speeds of free gravity waves, and does not reproduce the important property $|\omega| < N$ (see (17) of L13).
- AA has neither of these disadvantages

We conclude that *the AA is a better way of removing acoustic waves than the HA is.* The AA filters acoustic waves more effectively, and damages the gravity waves less.

This conclusion is mitigated (but not reversed) by 3 factors:

1. Since Lamb waves travel horizontally, their presence in hydrostatic models does not incur such a computational penalty as if vertically propagating acoustic waves had been present; this is because the horizontal grid spacing in a numerical model of atmospheric flow is usually much larger than the vertical grid spacing.
2. As we saw in L12, the AA requires a further approximation if consistent energetics are to be preserved: we must neglect a certain term in the vertical momentum equation at the same time as the $\partial\rho/\partial t$ term in the continuity equation.
3. Our analysis has been entirely in terms of linearised forms of the equations, whereas in practice you have to apply the results to the equations in their original, nonlinear form. This is not in principle difficult, but it turns out to be more complicated for the AA than the HA; we haven't time to go into the details.

Of these three factors: 1 is a stroke of luck for the HA; 2 is of theoretical interest but is unimportant in practice because the omitted term is nearly always negligible; and 3 explains why the HA has tended to be more popular amongst meteorological modellers than the AA has.

14. The effects of frame rotation: Poincaré, Kelvin and inertio-gravity waves

If we think in terms of velocities and accelerations measured relative to a rotating frame, we must remember that Newton's 2nd Law (N2) refers to velocities and accelerations measured in inertial (non-accelerating) frames. The transformation from an inertial frame to a rotating frame gives rise to Coriolis and centrifugal accelerations when we write N2 in terms of quantities measured in the rotating frame:

$$\left.\frac{d}{dt}\right|_{rot}\mathbf{u}_{rot} + \underbrace{2\Omega\times\mathbf{u}_{rot}}_{\text{Coriolis accel.}} + \underbrace{\Omega\times(\Omega\times\mathbf{r})}_{\text{Centrifugal accel.}} = \left.\frac{d}{dt}\right|_{ine}\mathbf{u}_{ine} = \underbrace{\mathbf{F}}_{\text{Force per unit mass}} \qquad (1)$$

The centrifugal acceleration, after a sign change, is combined with the force of true gravitational gravity to give *apparent* gravity acting in the direction of *apparent* vertical (which are what people with pendulums and plumb-lines measure). The Coriolis acceleration (or force, if we move it onto the r.h.s. of (1)) appears in the vertical and horizontal components of N2, and as $2\Omega\cos l$ as well as $2\Omega\sin l$, but its major influence is in the horizontal components of N2 as $f = 2\Omega\sin l$. The latitude (l) variation of f is the physical origin of Rossby waves, which we shall discuss briefly in L17. Here we examine the consequences of including f in various wave problems on the assumption that its latitude variation may be neglected (the "f-plane").

Linearised shallow water equations

The shallow water equations with Coriolis terms are

$$\frac{Du}{Dt} - fv + g\frac{\partial h}{\partial x} = 0; \quad \frac{Dv}{Dt} + fu + g\frac{\partial h}{\partial y} = 0; \quad \frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0 \qquad (1)$$

in which $h = h(x,y,t)$ is the depth of the layer. Linearising about a state of rest and uniform depth H ($h' = h - H$, but omitting the velocity component primes) gives:

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial h'}{\partial x} = 0; \quad \frac{\partial v}{\partial t} + fu + g\frac{\partial h'}{\partial y} = 0; \quad \frac{\partial h'}{\partial t} + H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0 \qquad (2)$$

Geostrophic flow

One solution of (2) is very simple. If $\partial/\partial t = 0$, then u and v must be in geostrophic balance with the height field, and the continuity equation is also satisfied

$$u = -\frac{g}{f}\frac{\partial h'}{\partial y}, \quad v = \frac{g}{f}\frac{\partial h'}{\partial x} \quad \Rightarrow \quad \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0$$

Thus, if f is constant, any flow in geostrophic balance is a solution of the linearised shallow water equations. To find something more interesting, consider $\partial/\partial t \neq 0$.

Poincaré waves

In terms of relative vorticity $\zeta \equiv (\partial v/\partial x - \partial u/\partial y)$ and divergence $D \equiv (\partial u/\partial x + \partial v/\partial y)$, (2) can be written (after cross differentiation) as

$$\frac{\partial \zeta}{\partial t} + fD = 0; \quad \frac{\partial D}{\partial t} - f\zeta + g\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)h' = 0; \quad \frac{\partial h'}{\partial t} + HD = 0 \qquad (3)$$

Differentiating the $\partial D/\partial t$ equation w.r.t. t, and using the other two equations of (3) for

$\partial\zeta/\partial t$ and $\partial h'/\partial t$ then gives: $$\left(\frac{\partial^2}{\partial t^2} + f^2 - gH\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right)D = 0 \qquad (4)$$

Solutions of (4) having $D \propto \exp[i(kx + ly - \omega t)]$ obey the dispersion equation

$$\omega^2 = f^2 + gH(k^2 + l^2) \qquad (5)$$

These waves are evidently shallow water waves modified by frame rotation ($f \neq 0$); a non-dispersive shallow water wave when $f = 0$ becomes dispersive when $f \neq 0$. They are known as *Poincaré* waves (although this term is sometimes reserved for the modes which obey the channel boundary conditions $v = 0$ at $y = 0,\ y_0$).

Exercises: 1. Derive a p.d.e. for h' from (3). Why isn't it quite the same as (4) ?
2. Show that the group velocity and phase velocity of Poincaré waves are parallel.
3. What is the condition for rotation effects to be negligible in (5) ?

Kelvin waves

If a *vertical* barrier is present (as in idealised ocean basins), and $f \neq 0$, then the shallow water equations can support an edge wave called the Kelvin wave which has a remarkable hybrid character. Suppose that $y = 0$ is a rigid barrier. Then there is a solution of (2) which has $v = 0$ everywhere, and u, h' which obey the remains of (2):

$$\frac{\partial u}{\partial t} + g\frac{\partial h'}{\partial x} = 0; \qquad \frac{\partial h'}{\partial t} + H\frac{\partial u}{\partial x} = 0; \qquad fu + g\frac{\partial h'}{\partial y} = 0 \qquad (6)$$

The first two of these equations lead to 1D surface gravity wave behaviour as if $f = 0$! Allowing amplitude variation with y we have

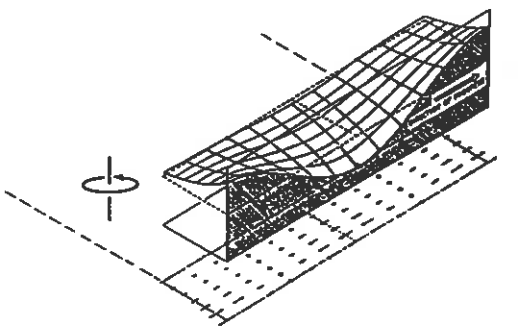
$$u = F(y)\exp[ik(x - ct)] \quad \text{and} \quad h' = (c/g)\,F(y)\exp[ik(x - ct)], \quad \text{with} \quad c = \pm\sqrt{gH} \qquad (7)$$

The third of equations (6) then requires

$$\frac{dF}{dy} + \frac{f}{c}F = 0 \qquad (8)$$

From (8): $$F \propto \exp[-fy/c] \qquad (9)$$

So $|F| \to \infty$ **either as** $y \to +\infty$ **or as** $y \to -\infty$, depending on whether the positive or the negative root is chosen in (7). If $c = +\sqrt{gH}$, F decays exponentially with increasing y, so the wave is permissible so long as the fluid is on the $y > 0$ side of the barrier. If $c = -\sqrt{gH}$, the wave is permissible so long as the fluid is on the $y < 0$ side of the barrier. Such waves are called *Kelvin waves*. They are non-dispersive, travel with the barrier on their right (in the NH), are trapped close to the barrier, have the character of surface gravity waves in the direction of travel and are in transverse geostrophic balance, with no transverse velocity component.



Exercises: 1. Why must Kelvin waves have the barrier on their right (in the NH) ?
2. Repeat the Kelvin wave problem with $f = \beta y$ ("equatorial β-plane") in (2) and (6).

Inertio-gravity waves

Any of the wave problems addressed in earlier lectures can be repeated with Coriolis terms present. The main effect is to increase the occurrence and extent of dispersion. For example, the approximate gravity wave dispersion equation (16) of L13 becomes

$$\omega^2 \approx \omega_g^2 = \left(N^2k^2 + f^2\left(m^2 + 1/4\overline{H}^2\right)\right)\Big/\left(k^2 + m^2 + 1/4\overline{H}^2 + f^2/c_s^2\right) \qquad (10)$$

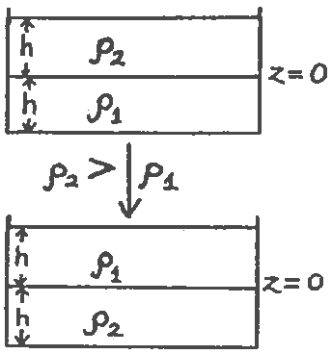
which describes *inertio-gravity* waves. Kelvin modes are possible in the presence of a vertical boundary (or close to the equator) but no other new types of wave are found.

15. Growing waves: the Kelvin-Helmholtz and Rayleigh-Taylor problems

All the waves we have met so far have been neutral; their amplitudes are constant in time. Waves whose amplitude change with time, and in particular growing waves, are the concern of this lecture. We begin our discussion by considering energy arguments and the association with instability. We note the results of the Kelvin-Helmholtz instability problem and then look at Rayleigh-Taylor instability in some detail.

Energetics

If growing waves are to exist in a system there must be a way of financing them: energy must be available, and there must be a way of converting this available energy into the energy of the wave. As a simple example, consider the arrangement of incompressible fluids shown. If $\rho_2 > \rho_1$, it is clearly "top-heavy", and we expect it to be unstable to small perturbations – in the absence of viscosity and interfacial tension, at least. We can easily make this notion more precise. Take $z = 0$ at the interface, and calculate the potential energy (PE) before and after exchange of the top and bottom fluids:

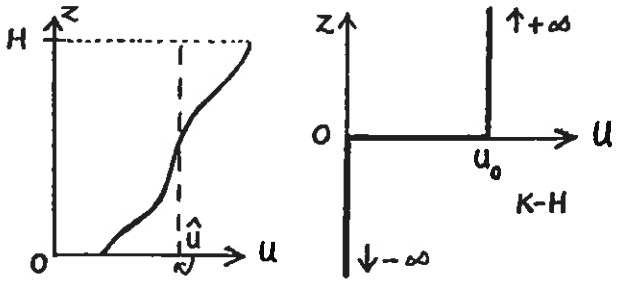


$$PE_{before} = \rho_2 g \int_0^H z\,dz - \rho_1 g \int_{-H}^0 z\,dz = \frac{gH^2(\rho_2 - \rho_1)}{2}$$

Clearly, $PE_{after} = -PE_{before}$; so the *reduction* of PE on exchange of the fluids is $gH^2(\rho_2 - \rho_1)$. This energy is available to perturbations applied to the initial state.

Shear instabilities; the Kelvin-Helmholtz problem

The source of energy for a growing disturbance may be kinetic rather than potential. Suppose that $U = U(z)$ for $0 < z < H$ (see diagram). If, somehow, we smooth out the flow so that it is uniform and equal to $\hat{U}$, then x-momentum will have been conserved if $\hat{U} = \frac{1}{H}\int_0^H U dz$.



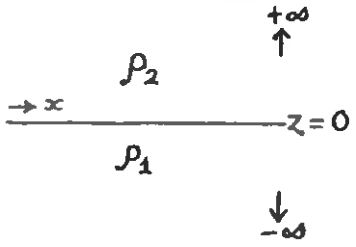
A reduction in kinetic energy (KE) is associated with this momentum-conserving smoothing of the flow (except in the special case in which U was uniform to start with). To show this, put $U = \hat{U} + U^*$:

$$\int_0^H U^2 dz - \int_0^H \hat{U}^2 dz = \int_0^H \left[\left(\hat{U} + U^*\right)^2 - \hat{U}^2\right] dz = \int_0^H \left[2\hat{U}U^* + \left(U^*\right)^2\right] dz = \int_0^H \left(U^*\right)^2 dz > 0$$

Thus any sheared flow represents a reservoir of kinetic energy. To find whether this energy can be released (and contribute to the energy of perturbations) we may carry out a linearised perturbation analysis. In the *Kelvin-Helmholtz* problem, fluid 1 is at rest (in $z < 0$), and fluid 2 ($z > 0$) has uniform velocity U_0 in the x direction; in the simplest version of the problem the fluids are homogeneous, incompressible and have the same density, and there is no viscosity or interfacial tension. It can be shown that the interface in this case is unstable to small perturbations of any wavelength (i.e. all wavelengths grow) and more unstable the shorter the wavelength of perturbation.

Rayleigh-Taylor instability

Consider the arrangement illustrated in the diagram: an incompressible liquid of density ρ_2 overlies a liquid of density ρ_1, the interface being at $z = 0$; the top fluid extends to $+\infty$, the lower to $-\infty$. Perturbations are applied to the interface, and surface tension T provides a stabilising influence if $\rho_2 > \rho_1$. This problem is an extension of the surface gravity wave problem we considered in L6. The dispersion relation for interfacial perturbations of the form $\eta = h\cos(kx - \omega t)$ turns out to be [see Annex]:



$$\omega^2 = \frac{k}{(\rho_1 + \rho_2)}\left\{Tk^2 + g(\rho_1 - \rho_2)\right\} \qquad (1)$$

Eq (1) has many aspects.

(a) If $T = \rho_2 = 0$, then $\omega^2 = gk$; the deep-water gravity wave dispersion relation (L6)

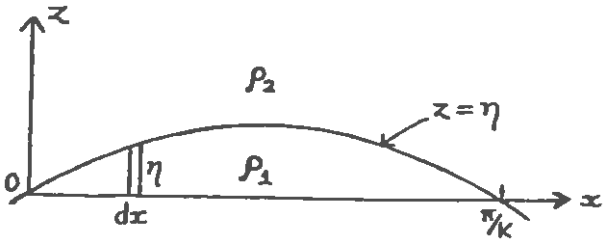
(b) If $T = 0$, then $\omega^2 = \frac{(\rho_1 - \rho_2)}{(\rho_1 + \rho_2)}gk$: the "reduced gravity" version of $\omega^2 = gk$

(c) If $T \neq 0$ and $\rho_2 = 0$, then (2) becomes $\omega^2 = \frac{Tk^3}{\rho_1} + gk$, which is the dispersion relation for *capillary waves*. We haven't time to discuss these!

(d) In (a) – (c), $\omega^2 > 0$ if $\rho_2 < \rho_1$, so the waves are neutral. Even if $\rho_2 > \rho_1$, (1) gives $\omega^2 > 0$ (neutral waves) if $Tk^2 > g(\rho_2 - \rho_1)$. If $Tk^2 < g(\rho_2 - \rho_1)$, however, $\omega^2 < 0$; **growing waves exist**, having amplitudes increasing as $\exp\left[\left(-\omega^2\right)^{1/2} t\right]$. If $T = 0$, all wavenumbers grow. If $T \neq 0$, only wavenumbers less than $\sqrt{g(\rho_2 - \rho_1)/T}$ grow; higher wavenumbers are neutral.

Energy argument to account for the stability threshold $\sqrt{g(\rho_2 - \rho_1)/T}$

Consider a half-wavelength of a perturbation $\eta = h\cos kx$ of the interface, as shown, and assume $\rho_2 > \rho_1$. The PE (per unit length in the y direction) of that body of fluid 1 which is above the undisturbed interface level ($z = 0$) is $\frac{\rho_1 g}{2}\int_0^{\pi/k} \eta^2 dx$.



The reduction in PE (compared to the undisturbed state) is therefore

$$\frac{\pi}{4k}(\rho_2 - \rho_1)gh^2. \qquad (2)$$

But work has been done against surface tension to increase the interface area (per unit length in the y direction) from π/k to $\int_0^{\pi/k}\left(1+(\partial\eta/\partial x)^2\right)^{1/2} dx$. Put $\eta = h\cos kx$ and expand the integrand for $h^2k^2 << 1$ to show that the increase in interface area is $(\pi/4k)h^2k^2$. So the work done against surface tension is $(\pi/4k)Th^2k^2$ (3)

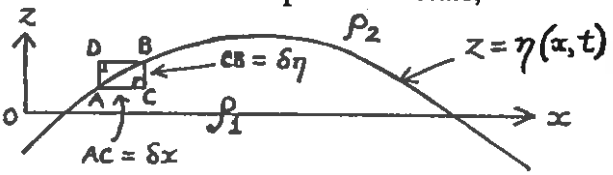
Results (2) and (3) suggest that wave growth will occur if $(\rho_2 - \rho_1)g > Tk^2$.

This agrees with the stability threshold wavenumber $\sqrt{g(\rho_2 - \rho_1)/T}$ found at (d).

15 (Annex) (i) Conditions at the interface between two incompressible fluids (ii) Analysis for the Rayleigh-Taylor instability problem

Both in the surface wave problem (L6) and the Rayleigh-Taylor problem (L15) we apply boundary conditions at an interface. In the (idealised) surface wave problem the interface is between an incompressible liquid and a vacuum, but this is a limiting case in which the density of the top liquid vanishes. This Annex sets out the reasoning behind the various continuity conditions more carefully than we cared to do in L6; in particular, the conditions on the pressure field are clarified. The results are then applied to the Rayleigh-Taylor problem.

Incompressible liquid 2 (density ρ_2) overlies incompressible liquid 1 (density ρ_1), and we choose $z = 0$ at the unperturbed horizontal interface. In the perturbed state, the interface is at $z = \eta(x,t)$; we consider only interface displacements that are independent of the transverse horizontal coordinate y. Let subscript j refer to fluid j.



Pressure continuity conditions

Consider a very small section AB of the interface. We can reach B from A by moving within fluid 1 horizontally to point C and then vertically to B, or by moving within fluid 2 vertically to point D and then horizontally to B. AC = DB = δx; AD = CB = $\delta\eta$. The pressure change between A and B must be the same whether we choose to go via C or D (or by any other route, for that matter):

$$\delta p_{AB} = \delta p_{AC} + \delta p_{CB} = \delta p_{AD} + \delta p_{DB}$$

Hence

$$\delta p_{AB} = \frac{\partial p_1}{\partial x}\delta x + \frac{\partial p_1}{\partial z}\delta\eta = \frac{\partial p_2}{\partial x}\delta x + \frac{\partial p_2}{\partial z}\delta\eta$$

Thus we require

$$\frac{\partial p_1}{\partial x} + \frac{\partial p_1}{\partial z}\frac{\partial \eta}{\partial x} = \frac{\partial p_2}{\partial x} + \frac{\partial p_2}{\partial z}\frac{\partial \eta}{\partial x} \quad \text{at } z = \eta \qquad (1)$$

The p_j are the total pressures: $p_1 = \bar{p}_1(z) + p_1'(x,z,t)$; $p_2 = \bar{p}_2(z) + p_2'(x,z,t)$

with $\partial\bar{p}_1/\partial z = -\rho_1 g$

and $\partial\bar{p}_2/\partial z = -\rho_2 g$

Thus

$$\frac{\partial p_1'}{\partial x} + \left(\frac{\partial p_1'}{\partial z} - \bar{\rho}_1 g\right)\frac{\partial \eta}{\partial x} = \frac{\partial p_2'}{\partial x} + \left(\frac{\partial p_2'}{\partial z} - \bar{\rho}_2 g\right)\frac{\partial \eta}{\partial x} \quad \text{at } z = \eta \qquad (2)$$

Since the interface displacement η is itself a perturbation quantity, the linearised form of (2) is

$$\frac{\partial p_1'}{\partial x} - \bar{\rho}_1 g\frac{\partial \eta}{\partial x} = \frac{\partial p_2'}{\partial x} - \bar{\rho}_2 g\frac{\partial \eta}{\partial x} \quad \text{at } z = 0 \qquad (3)$$

Eq(3) may be applied either as it stands or in the integrated form

$$p_2' - p_1' = (\rho_2 - \rho_1)g\eta \qquad (4)$$

To take account of the effects of interfacial tension, T, we may add a term $T\partial^2\eta/\partial x^2$ to the r.h.s. of (4):

$$p_2' - p_1' = (\rho_2 - \rho_1)g\eta + T\frac{\partial^2\eta}{\partial x^2} \qquad (5)$$

Velocity continuity conditions

The two fluids must remain in contact. Thus their velocities normal to the interface, at any given point on the interface, must be the same. [In the absence of friction, their velocities tangential to the interface need not be the same.] The velocity normal to the interface is $\mathbf{u}\cdot\mathbf{n}$ (where $\mathbf{u}$ is the velocity and $\mathbf{n}$ is the unit normal) and this quantity must have the same value on the interface in each fluid. A vector proportional to $\mathbf{n}$ is $(-\partial\eta/\partial x, 1)$; so, since $\mathbf{u}_i = (u_i', w_i')$, we require

$$w_1' - u_1'\frac{\partial\eta}{\partial x} = w_2' - u_2'\frac{\partial\eta}{\partial x} \qquad \text{at } z = \eta \qquad (6)$$

Once again recognising the perturbation character of the interface displacement η, we obtain the linearised version of (6) as

$$w_1' = w_2' \qquad \text{at } z = 0 \qquad (7)$$

The vertical velocity at the interface must also obey the linearised version of $w = D\eta/Dt$ at $z = \eta$, i.e. $w = \partial\eta/\partial t$ at $z = 0$. Hence we require

$$w_1' = w_2' = \partial\eta/\partial t \qquad \text{at } z = 0 \qquad (8)$$

Analysis for the Rayleigh-Taylor problem

If the interface has the wavy form $\eta = h\cos(kx - \omega t)$, then [from (5)] the pressure perturbations p_j' at $z = 0$ must have the same form: $p_j'(0) = P_j\cos(kx - \omega t)$. (9)

Putting (9) in (5) gives $P_2 - P_1 = h\left[g(\rho_2 - \rho_1) - Tk^2\right]$. (10)

The $p_j'(z)$ obey Laplace's equation (see L5), and remain finite as $z \to -\infty$ $(j = 1)$ and as $z \to +\infty$ $(j = 2)$. Thus

$$p_2'(z) = P_2\exp(-kz)\cos(kx - \omega t) \qquad (11)$$

$$p_1'(z) = P_1\exp(+kz)\cos(kx - \omega t) \qquad (12)$$

As in the surface wave problem (L6) we now use the vertical component of the momentum equation to find $w_j'(z)$:

$$\rho_j\frac{\partial w_j'}{\partial t} = -\frac{\partial p_j'}{\partial z} \quad \Rightarrow \quad \begin{cases} \rho_2 w_2' = -(kP_2/\omega)\exp(-kz)\sin(kx - \omega t) & (13) \\ \rho_1 w_1' = +(kP_1/\omega)\exp(+kz)\sin(kx - \omega t) & (14) \end{cases}$$

Applying (8) now gives $P_1\rho_2 + P_2\rho_1 = 0$ (15)

and $\omega^2\rho_2 h + kP_2 = 0$ (16)

From (10) and (15):

$$P_2(\rho_2 + \rho_1) = \rho_2 h\left[g(\rho_2 - \rho_1) - Tk^2\right] \qquad (17)$$

Finally, from (16) and (17):

$$\omega^2 = \frac{k}{(\rho_1 + \rho_2)}\left\{Tk^2 - g(\rho_2 - \rho_1)\right\}$$

16. Waves in inhomogeneous basic states

Most of the wave analyses of earlier lectures involved simple differential equations with constant coefficients – because we chose simple basic states. Here we examine what happens if more realistic basic states are chosen; we outline various results and techniques, using as our example the 2D problem of steady incompressible flow over mountain ridges (begun in L10, 11). This problem does not allow us to study the flow over (or around?) isolated mountains, or the effects of changes of basic state wind direction with height, but it gives useful insight. Maybe this is the time to recall why meteorologists are interested in such problems: by studying idealised systems you develop a feeling for key effects and parameters in the real atmosphere, for the range of possible behaviour, and for its sensitivity to changes of parameter values and other assumed conditions. This helps you to make sense of data, and to design and diagnose numerical models of the atmosphere more effectively. Progress depends on observation, theory and numerical modelling being done in harmony.

Flow over isolated ridges; $\overline{U}$ and N independent of height

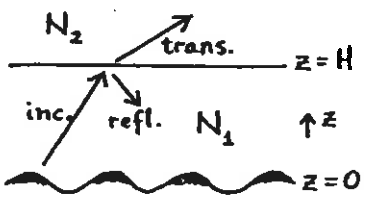
In L11 we noted that the results obtained in L10 assuming sinkx corrugations of the lower boundary could be extended to other profiles by Fourier synthesis. A much studied example of this procedure depends on a certain definite integral:

$$F(x;L) \;=\; L\int_0^\infty e^{-kL}\cos kx\,dk \;=\; \Re\int_0^\infty Le^{-k(L-ix)}dk \;=\; \Re\left\{\frac{L}{L-ix}\right\} \;=\; \frac{1}{1+(x/L)^2}$$

[L is any real constant; $\Re$ means "real part of".] Suppose we solve the problem of L10 for $h = h_M\cos kx$, and find $w' = w_k(x,z)$. Then we can find the solution for the bell-shaped ridge $h = h_M/\left[1+(x/L)^2\right]$ by forming $L\int_0^\infty e^{-kL}w_k(x,z)dk$. Results obtained by Queney are amongst those shown on the handout **"L16. Diagrams"** (all from Gill's book). The response in the hydrostatic case (Fig. 8.9b) is directly above the hill, whereas in the nonhydrostatic case (Fig. 8.9a) it extends downstream somewhat. In both cases, the response extends above 10 km. Queney's analysis assumes $\partial\overline{U}/\partial z = \partial N/\partial z = 0$; if $\overline{U}$ and N vary with z, the response can be very different.

Layer models representing height variation of $\overline{U}$ and N

One way of representing height variations is to deal with layers within each of which $\overline{U}$ and N are constant. Then the general solutions within each layer are familiar, and we only have to match w and p properly at the interfaces in order to derive the solution we seek. For example, we might use just 2 layers, with different values of N but the same values of $\overline{U}$. If the layer interface is at $z = H$, the response to flow over sinusoidal ridges at $z = 0$ consists of a *transmitted* (and refracted) wave above $z = H$, and an *incident* wave and a *reflected* wave below $z = H$. As you might expect, the solution depends on the height of the interface in relation to the wavelength of the mountains. Partial, or "leaky" resonances can occur; the resonances will never be total because some wave energy always goes into the upper layer. Fig. 8.14 shows an example. See Fig. 8.13 for a real case of severe downslope winds (albeit on a larger scale).



The upper boundary condition in numerical models

An extreme case of an interface is a rigid lid (at which $w = 0$); then there is no transmitted wave and all the energy of the incident wave is reflected. We can investigate the consequences for the problem we did in L10. With $h = h_0 e^{ikx}$,

$$m^2 = \frac{N^2}{\overline{U}^2} - k^2 \quad \text{and} \quad w'(0) = \overline{U} k h_0 e^{ikx}; \quad \text{but now} \quad w'(H) = 0.$$

From the general solution $w'(z) = \left(Ae^{imz} + Be^{-imz}\right)e^{ikx}$ and the conditions on w' we find that

$$w'(z) = \overline{U} i k h_0 \frac{\sin m(H-z)}{\sin mH} e^{ikx}.$$

This is quite unlike the solution obtained in L10. Resonances can even occur. It is very important to avoid spurious reflections from boundaries in numerical models!

Linearised steady wave problem when $\overline{U}$ and N are given functions of height

When $\overline{U} = \overline{U}(z)$ and $N = N(z)$, the linearised equations (1) – (4) of L10 contain only one further term – in Eq (1), which becomes

$$\overline{U}\frac{\partial u'}{\partial x} + w'\frac{d\overline{U}}{dz} + \frac{\partial}{\partial x}\left(\frac{p'}{\overline{\rho}}\right) = 0 \qquad (1^*)$$

It is readily shown (*Exercise?*) from (2) – (4) of L10, and (1*), that w' obeys

$$\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2} + \left[\frac{N^2}{\overline{U}^2} - \frac{1}{\overline{U}}\frac{d^2\overline{U}}{dz^2}\right]w' = 0. \qquad (2)$$

The only "new" term in (2) [cf. (5) of L10] is the one involving $d^2\overline{U}/dz^2$. The quantity in square brackets in (2) is called the Scorer parameter; its height variation determines the solution. In solving (2) one can "layerise" as described above, but with the Scorer parameter being the quantity that is constant within each layer. A 2-layer solution, obtained by Scorer, is shown in Fig. 8.11; wave trapping below about 3 km is evident, and downstream extent is considerable. One can of course solve (2) numerically, but it is always advisable to have some analytical apparatus to hand.

Ray tracing; critical levels

In the ray-tracing approach (named from geometrical optics) group velocity formulae calculated for the case in which $\overline{U}$ and N are both constant (as in L10) are used to calculate the slopes of *ray paths* $z = z(x)$ which are the trajectories of wave energy:

$$dz/dx = c_{gz}/c_{gx} \quad \text{where} \quad \mathbf{c}_g = \left(c_{gx}, c_{gz}\right) \qquad (3)$$

Fractional changes in $\overline{U}$ and N are assumed small in a WKB sense, i.e. over a vertical wavelength; the height variations of $\overline{U}$ and N are then restored during the integration of (3) for $z(x)$. The vertical wavenumber m given by $m^2 = N^2/\overline{U}^2 - k^2$ (as in L10) is a key quantity [horizontal wavenumber k is held constant for each ray]. Two special situations may arise. (i) $\underline{m \to 0}$. The solution is about to change from propagating to trapped. In the internal gravity wave case, the ray attains a horizontal direction and then turns back; Fig. 8.12a shows an example. (ii) $\underline{m \to \infty}$. Corresponds to $\overline{U} \to 0$, the approach to a *critical level* – a level at which the basic flow speed is equal to the wave phase speed (zero for steady waves). In the internal gravity wave problem, c_{gz} becomes very small (as m becomes very large) and absorption of wave energy occurs; see Fig. 8.12b. An experimental demonstration of wave confinement below a critical level is shown in Fig. 8.15. Behaviour in other problems is still a matter of study.

L16. Diagrams

These are all taken from A.E.Gill's "Atmosphere-Ocean Dynamics", which gives a thorough and authoritative treatment of gravity waves. One caption has been re-typed to avoid confusing reference to parts of the Figure (8.9(c)) that I haven't copied.

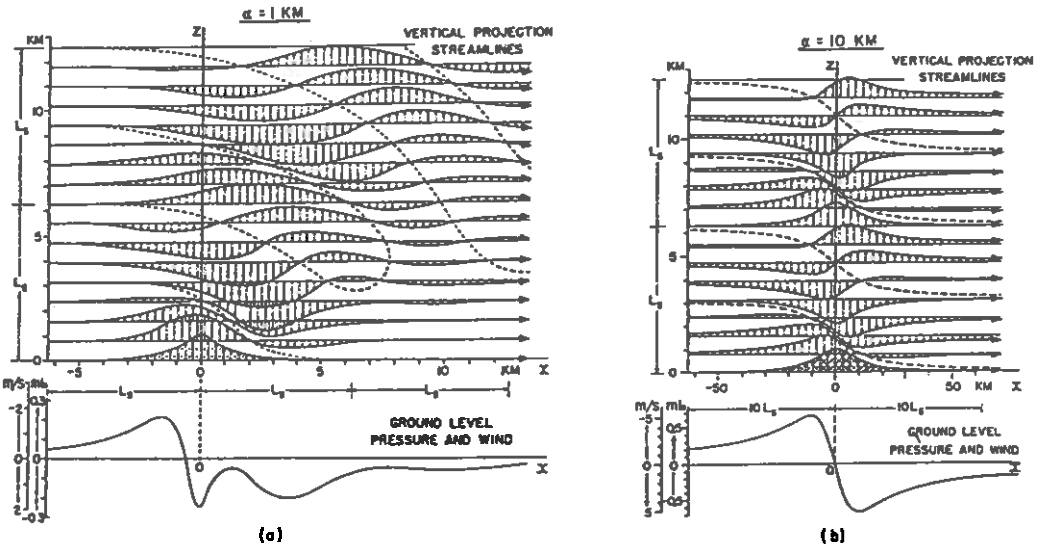


Fig. 8.9 Steady waves generated by flow with uniform velocity ($\bar{U} = 10\ \mathrm{ms}^{-1}$) of a uniformly stratified ($N = 0.01\ s^{-1}$) fluid over bell-shaped ridges of various widths L (from Queney 1948). The mountain profile is $h = h_M \big/ \left(1 + x^2/L^2\right)$ and the solutions are based on linear theory. Case (a) is for $L = U/N = 1$ km and typifies the nonhydrostatic wave regime. Case (b) is for $L = 10$ km and typifies the hydrostatic regime in which rotation is not important. The upper part of each diagram shows the trajectories of air particles. The dashed lines show where the vertical displacement is zero. The scale L_s is defined by $L_s = 2\pi\bar{U}/N$, and in both (a) and (b) is a good measure of the vertical wavelength found. The lower panels of (a) and (b) show the ground-level pressure and wind variations associated with the waves.

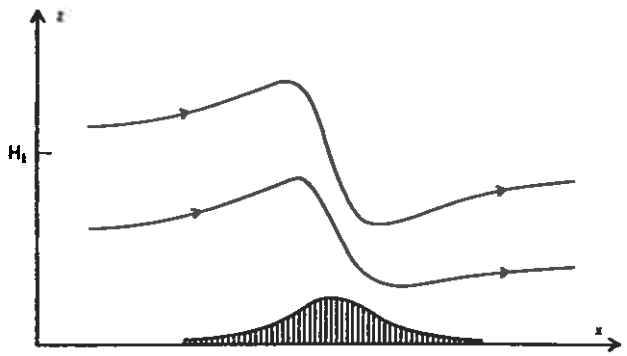
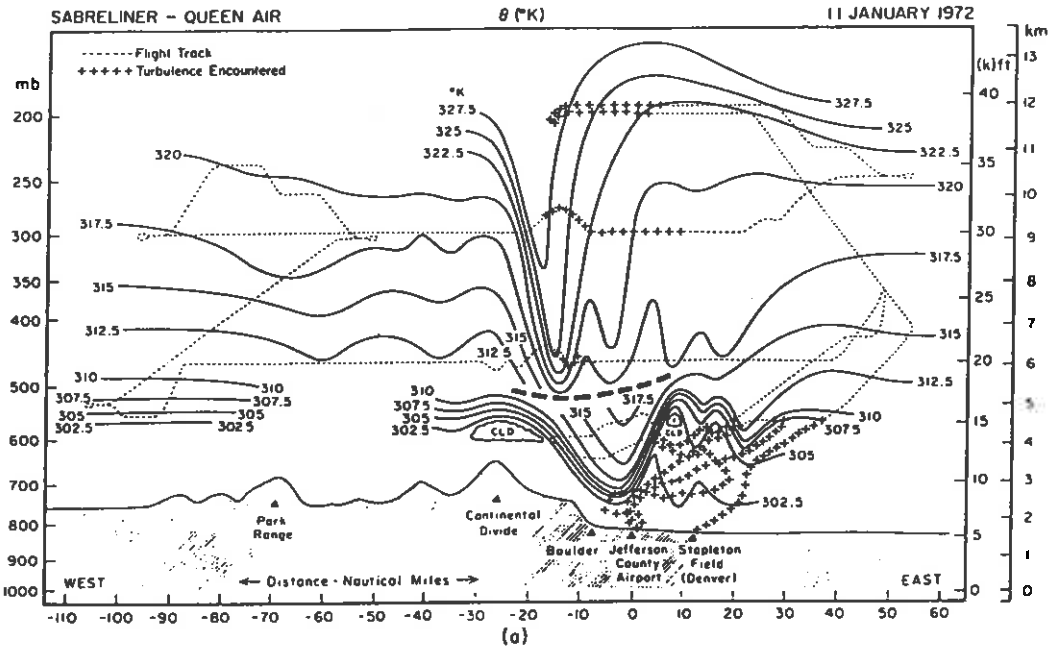


Fig. 8.14. Particle paths for air flow over a bell-shaped mountain, obtained for the linear hydrostatic non-rotating case. The undisturbed flow has uniform velocity U, constant buoyancy frequency N_1 up to height H_1, and constant buoyancy frequency $\frac{1}{3}N_1$ above H_1. The (partial resonance) condition for maximum response is satisfied, namely, $N_1 H_1/U = \pi/2$. Then a quarter of a wavelength is obtained between the ground and the discontinuity in buoyancy frequency, so waves reflected off the discontinuity and the ground reinforce each other. The resonance is only partial in the sense that only partial reflection occurs at the discontinuity. Note that the particle displacements at the discontinuity are significantly greater than at the ground. The pattern at higher levels is the same as that shown in Fig. 8.9b for uniform conditions. Note that the displacements are much larger than the height of the mountain, whereas for constant N (Fig. 8.9b) they are comparable.



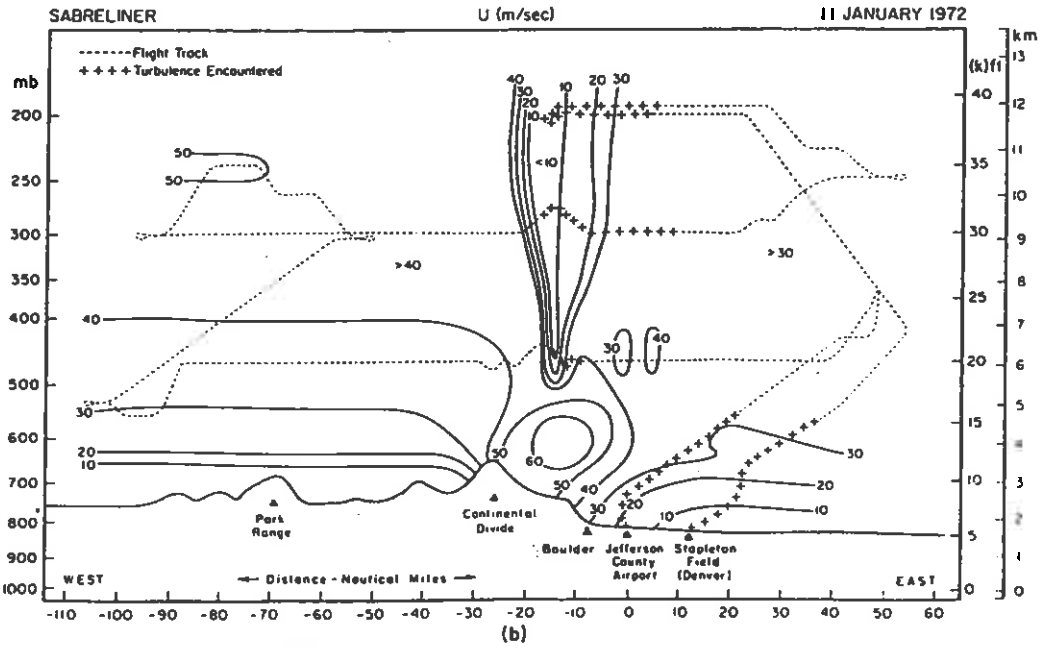


Fig. 8.13. Observations of (a) potential temperature (in degrees Kelvin) and (b) velocity (in meters per second) in a section across the Rockies in Colorado on January 11, 1972. The dashed lines show aircraft tracks with periods of significant turbulence shown by plus signs. The heavy dashed line separates observations made a few hours apart—it is quite possible that at a given time, the upper dip in the potential temperature contours was directly above the lower one. [From Lilly 1978, Figs. 7 and 9).]

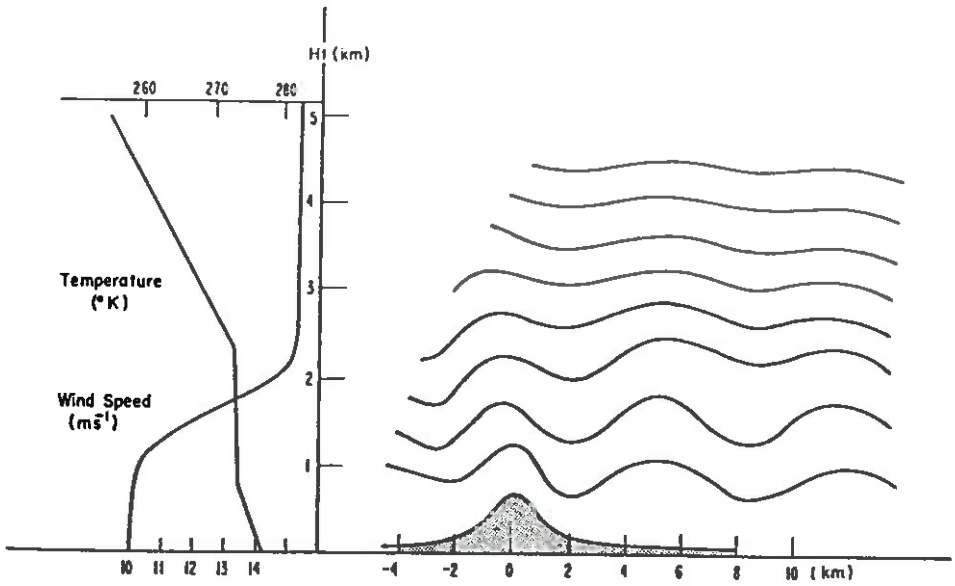


Fig. 8.11. Air flow over a bell-shaped mountain of width L = 1 km when the wind and temperature structure is as shown in the left panel. The result is that of Scorer (1949) as corrected by Gossard and Hooke (1975, Fig. 57-1). The vertical structure is such that it strongly emphasizes waves with a horizontal wavelength of 5.5 km, and a regular train of these waves can be seen in the lee of the mountain.

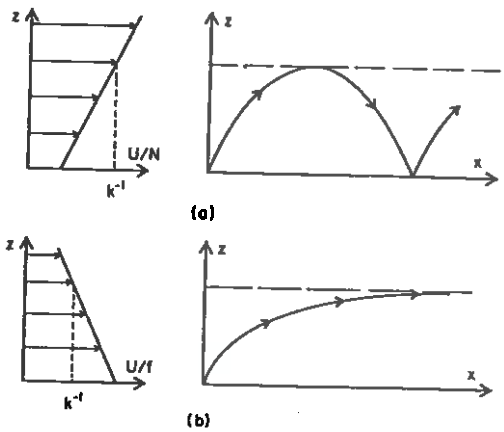
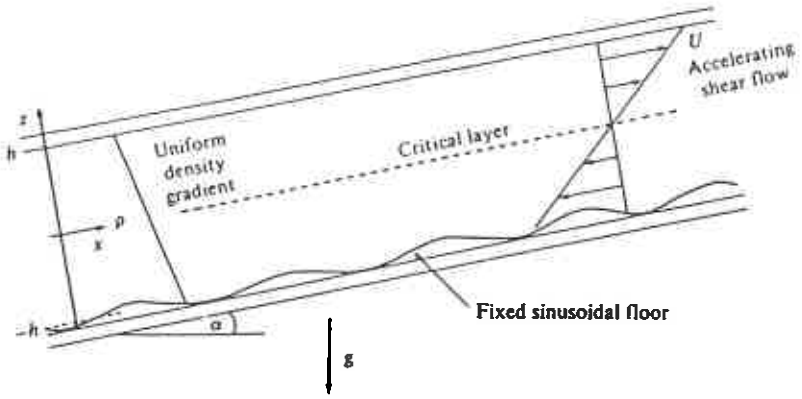
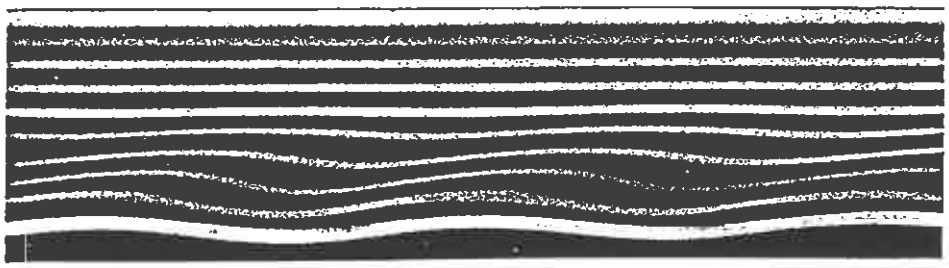


Fig. 8.12. Special types of ray paths for a medium whose properties vary slowly with vertical coordinate z. The paths are those that would be followed by a hypothetical particle traveling at the local group velocity $\mathbf{c}_g$, this being calculated as a function of U and N, using the formula for a uniform medium. The horizontal component k of the wavenumber and frequency ω are fixed for each ray. (a) Case exhibiting wave reflection. This occurs for nonhydrostatic waves when U/N increases with height and exceeds k^{-1}. The waves refract as shown and become horizontal at the level at which $Uk = N$. Thus wave energy can be trapped in a layer near the ground. (b) Case exhibiting absorption. This occurs for hydrostatic waves with wavenumbers such that rotation effects are important when U decreases with height and drops to a value equal to $|f|/k$. Upward-propagating waves approach this level asymptotically. The vertical component of the wavenumber continually decreases, so friction effects cause the wave to dissipate.



(a)



(b)

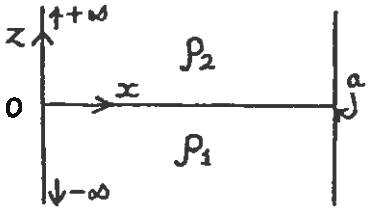
Fig. 8.15. An experimental demonstration of critical layer absorption of internal waves. (a) The apparatus—an inclined tube with a corrugated floor. (b) Waves observed in an accelerating shear flow over a corrugated floor of amplitude 0.5 cm and wavelength 25 cm. The tube contains stratified fluid with $N = 2.626\ s^{-1}$ and has been tilted through 5.2°. [From Thorpe (1981, Figs. 1 and 4e).]

17. Rossby waves, non-normal modes and barotropic instability

Our only adventure into waves in rotating fluids so far was in L14, where we discussed Poincaré and Kelvin waves and (briefly) inertio-gravity waves, assuming constant f. Here we examine some aspects of the barotropic vorticity equation when f varies with latitude, noting Rossby waves and some important consequences of a non-uniform basic flow. First we discuss a relevant variant of the Rayleigh-Taylor problem (L15).

Normal modes in a Rayleigh-Taylor problem

Suppose that the 2 incompressible fluids of the Rayleigh-Taylor problem are confined by rigid (but slippery) boundaries at $x = 0, a$. Then we must have $\partial u/\partial x = 0$ at $x = 0, a$;



hence $\partial p_2'/\partial x = \partial p_1'/\partial x = \partial \eta/\partial x = 0$ there. If we look for solutions $\eta = h_n \cos\left(\frac{n\pi x}{a}\right)\cos\omega_n t$ $(n = 1, 2, 3, ...)$ (1) (which obey the boundary conditions) then we find that

$$(\rho_1 + \rho_2)\omega_n^2 = \frac{n\pi}{a}\left[\frac{n^2\pi^2}{a^2}T - (\rho_2 - \rho_1)g\right] \qquad (2)$$

This is the dispersion relation found in L15 (Eq(1)), but with wavenumber k quantised as $k = n\pi/a$. Instability will *not* occur if $\frac{T\pi^2}{a^2} > (\rho_2 - \rho_1)g$). The functions $\cos(n\pi x/a)$ in (1) are associated with the frequencies ω_n given by (2); (1) and (2) specify the *normal modes* of the system; they are *normal* because the functions $\{\cos(n\pi x/a)\}$ form a complete, orthonormal set on the interval $x = [0, a]$ for all functions having zero gradient at $x = 0, a$:

$$\frac{2}{a}\int_0^a \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi x}{a}\right)dx = \delta_{mn} = \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases} \qquad (3)$$

If, at $t = 0$, we have $\eta = \eta_0(x)$, we can expand $\eta_0(x)$ as a Fourier cosine series,

$$\eta_0(x) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right), \text{ where } A_n = \frac{2}{a}\int_0^a \eta_0(x)\cos\left(\frac{n\pi x}{a}\right)dx.$$

The solution for $t \geq 0$ is then $\eta(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right)\cos\omega_n t$ (4)

The normal modes thus give the solution for all $t > 0$. This result holds for all the fields in the problem. We can associate fixed amounts of kinetic and potential energy with each mode because their spatial functions are orthonormal. Precisely the same situation arose in the normal mode problems we studied in examples at the beginning of the course.

The barotropic vorticity equation and Rossby waves

Consider the barotropic vorticity equation on a mid-latitude β-plane:

$$\frac{\partial}{\partial t}\nabla^2\psi + (\mathbf{v}_\psi \cdot \nabla)(\nabla^2\psi + \beta y) = 0 \qquad (5)$$

Here ψ is the streamfunction of the horizontal flow: $\mathbf{v}_\psi = (-\partial\psi/\partial y, \partial\psi/\partial x)$

Also, $\nabla = (\partial/\partial x, \partial/\partial y)$; $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$; and $\beta = df/dy$ is a constant.

If we linearise (5) about a *uniform* zonal flow $\overline{U} = U_0$ we readily find

$$\left(\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}\right)\nabla^2\psi' + \beta\frac{\partial\psi'}{\partial x} = 0 \qquad (6)$$

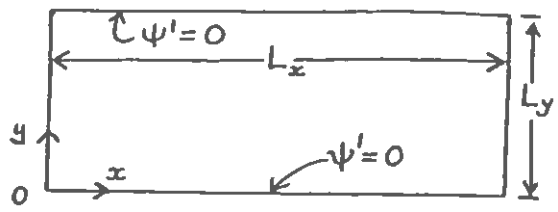
Let us require $\psi'(x + L_x) = \psi'(x)$ and $\psi'(x,0) = \psi'(x, L_y) = 0$ (7)

Conditions (7) are obeyed by the wavy forms

$$A_n \sin\left(n\pi y/L_y\right)\exp\left\{2\pi mi(x-ct)/L_x\right\}\ , \qquad (8)$$

which also satisfy (6) so long as

$$c \;=\; c_{mn} \;\equiv\; U_0 - \frac{\beta}{\left(k_m^2 + l_n^2\right)} \qquad (9)$$



(where $k_m = 2\pi m/L_x$ and $l_n = \pi n/L_y$). Eq(9) is the Rossby wave retrogression formula. The functions $\left\{\sin\left(n\pi y/L_y\right)\exp\left(2\pi mix/L_x\right)\right\}$ are orthonormal on the interval $y=\left[0, L_y\right]$, so we can write any initial ψ' as $\psi'(x,0) = \sum_{m=-\infty}^{m=+\infty}\sum_{n=1}^{\infty} \Psi_{mn} \sin\left(\frac{n\pi y}{L_y}\right)\exp\left\{\frac{2\pi mix}{L_x}\right\}$.

The solution for $t \geq 0$ is $\psi'(x,t) = \sum_{m=-\infty}^{m=+\infty}\sum_{n=1}^{\infty} \Psi_{mn} \sin\left(\frac{n\pi y}{L_y}\right)\exp\left\{\frac{2\pi mi}{L_x}\left(x - c_{mn}t\right)\right\}$. (10)

Now let us repeat this problem, but with a basic flow that is not uniform: $\overline{U} = \overline{U}(y)$

In place of (6) we find $\left(\frac{\partial}{\partial} + \overline{U}(y)\frac{\partial}{\partial x}\right)\nabla^2\psi' + \left(\beta - \frac{d^2\overline{U}}{dy^2}\right)\frac{\partial\psi'}{\partial x} \;=\; 0$ (11)

Eq (11) is clearly a more difficult equation to solve than (6). Rather less clearly, (11) is a *qualitatively different* equation from (6). It can be shown that (11) implies

$$\frac{d}{dt}\iint(\nabla\psi')^2\,dxdy \;=\; -2\iint\frac{d\overline{U}}{dy}\,\frac{\partial\psi'}{\partial x}\,\frac{\partial\psi'}{\partial y}\,dxdy. \qquad (12)$$

Here the integrations extend over $x = 0, L_x$ and $y = 0, L_y$, and we have applied (7). The l.h.s. of (12) gives the rate of change with time of the kinetic energy (KE) of the wave. Eq(12) shows that the total wave KE is conserved if $d\overline{U}/dy = 0$, but not otherwise (in general). Suppose we look for wavy solutions of (11) of the form $A(y)\exp\left\{ik_m(x-ct)\right\}$.

Then $A(y)$ must obey $\left(\overline{U}-c\right)\left(\frac{d^2A}{dy^2} - k_m^2A\right) + \left(\beta - \frac{d^2\overline{U}}{dy^2}\right)A \;=\; 0$ (13)

with $A(0) = A(L_y) = 0$. This problem has many solutions, each being an eigenfunction $A(y)$ associated with an eigenvalue c. Only when $d\overline{U}/dy = 0$ (as in (6)) are the eigenfunctions orthogonal, and is a fixed amount of KE associated with each mode. To overcome this difficulty we can re-define what we mean by energy and orthogonality. Two quantities quadratic in the perturbations that *are* conserved in the area average are

$$P \;\equiv\; \iint\frac{\left(\nabla^2\psi'\right)^2}{\left[\beta - d^2\overline{U}/dy^2\right]}dxdy \quad\text{and}\quad E \;=\; \iint\left\{(\nabla\psi')^2 - \frac{\overline{U}\left(\nabla^2\psi'\right)^2}{\left[\beta - d^2\overline{U}/dy^2\right]}\right\}dxdy$$

which are called the *pseudomomentum* (P) and *pseudoenergy* (E) by Held (J.A.S. 1985). This issue is beyond our reach here – but see the attached "L17: Supplementary notes".

Barotropic instability

Multiplication of (13) by $A^*/\left(\overline{U}-c\right)$, and some manipulation (see attached notes), leads to the simple result $c_i\int\left(\beta - \frac{d^2\overline{U}}{dy^2}\right)\frac{|A|^2}{\left|\overline{U}-c\right|^2}dy \;=\; 0$, in which c_i is the imaginary part of the eigenvalue c. Hence the basic state *absolute vorticity gradient* $\left(\beta - d^2\overline{U}/dy^2\right)$ must change sign if instability is to occur ; this is the Rayleigh-Kuo necessary condition for barotropic instability. A flow having shear is not necessarily unstable (cf. L15).

L17. Supplementary notes: pseudomomentum and pseudoenergy; a barotropic instability criterion; adjointness

These notes briefly describe some important results obtained by I. M. Held (*J. Atmos. Sci.*, **42**, 2280-2289; 1985). A derivation of the Rayleigh-Kuo necessary criterion for barotropic instability is also given, and finally some comments on adjointness.

1. *Time variations of global energy, enstrophy, pseudomomentum and pseudoenergy in a numerical integration of the linearised barotropic vorticity equation on a sphere*

The paper by Isaac Held (reference above) clearly describes how and why the notion of ascribing energy to modes collapses if the modes are not orthogonal. He carried out numerical integrations of the linearised barotropic vorticity equation (BVE) on a global domain, starting from a realistic zonal average flow (see Fig 1) with a zonal wave 3 disturbance superimposed in middle latitudes. A viscous term $\nu\nabla^4\psi$ was added to the r.h.s. of the BVE, ν being given the (small) value $1.0 \times 10^4\ m^2 s^{-1}$; this term mopped up energy reaching the smallest spatial scales, and led to a gradual decrease of the *total* energy (zonal flow + wave) in the model.

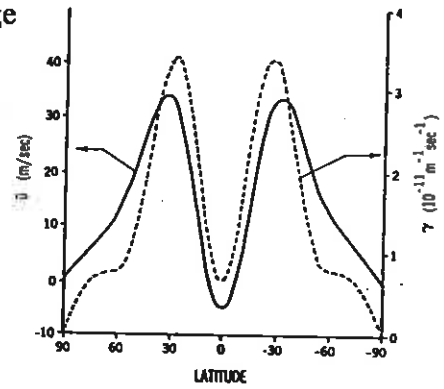


FIG. 1. The basic state zonal flow $\bar{u}$ and the corresponding absolute vorticity gradient γ used in the numerical example of barotropic decay on the sphere.

Held's plots of the *wave* (= "eddy") *enstrophy, pseudomomentum, energy* and *pseudoenergy* (Fig 2) show that the energy and enstrophy fluctuate with a time-scale of a day or so (period ~ 3 days), whereas the pseudomomentum and pseudoenergy do not show these fluctuations, exhibiting only the general trend which reflects the gradual viscous run-down of the circulation over 40 days of model integration. This behaviour is consistent with the theoretical arguments of Held's paper. Note that *enstrophy* is defined as the square of the relative vorticity, and that Fig 2 shows globally integrated values of each quantity.

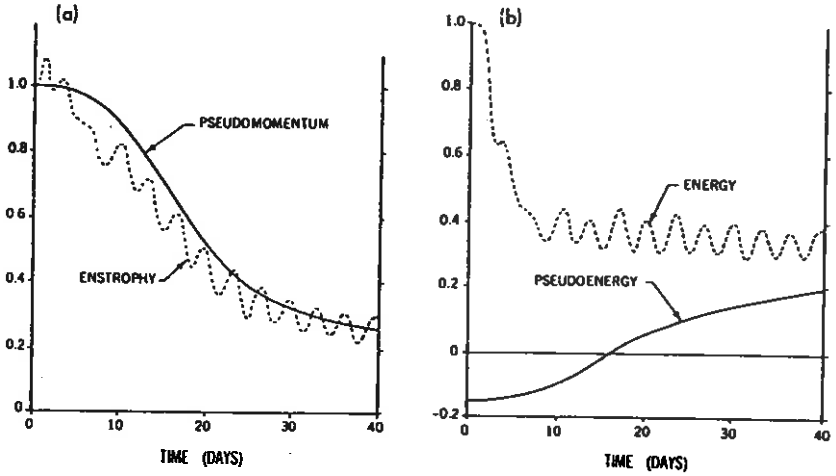


FIG. 2. (a) The time evolution of the globally integrated eddy enstrophy and pseudomomentum, both normalized to unity at $t = 0$, for the initial condition given in the text; (b) the time evolution of the integrated eddy energy and pseudoenergy, with the eddy energy normalized to unity at $t = 0$.

2. *Derivation of a necessary condition for barotropic instability*

Eq (13) can be written as

$$(\bar{U}-c)\left(\frac{d^2A}{dy^2}-k_m^2A\right) + \gamma A = 0 \qquad (14)$$

in which

$$\gamma = \gamma(y) \equiv \beta - d^2\bar{U}/dy^2 \qquad (15)$$

Multiply (14) by $A^*/(\overline{U}-c)$ (where A^* means the complex conjugate of A):

$$A^*\frac{d^2A}{dy^2} - k_m^2|A|^2 + \gamma\frac{|A|^2}{(\overline{U}-c)} = 0 \qquad (16)$$

Now $$A^*\frac{d^2A}{dy^2} = \frac{d}{dy}\left(A^*\frac{dA}{dy}\right) - \frac{dA^*}{dy}\frac{dA}{dy} = \frac{d}{dy}\left(A^*\frac{dA}{dy}\right) - \left|\frac{dA}{dy}\right|^2 \qquad (17)$$

and $$\frac{1}{(\overline{U}-c)} = \frac{\overline{U}-c^*}{|\overline{U}-c|^2} = \frac{\overline{U}-c_r+ic_i}{|\overline{U}-c|^2} \quad \text{(where } c = c_r + ic_i\text{)}. \qquad (18)$$

Use of (17) and (18), followed by integration over $y = 0, L_y$ and application of the boundary conditions $A(0) = A(L_y) = 0$, gives

$$\int\left\{\left|\frac{dA}{dy}\right|^2 + k_m^2|A|^2\right\}dy = \int\gamma(\overline{U}-c_r+ic_i)\frac{|A|^2}{|\overline{U}-c|^2}dy \qquad (19)$$

Separation of the imaginary part of (19), and recall of the definition of γ, gives

$$c_i\int\left(\beta-\frac{d^2\overline{U}}{dy^2}\right)\frac{|A|^2}{|\overline{U}-c|^2}dy = 0 \qquad (20)$$

So if c_i is non-zero, $\left(\beta-d^2\overline{U}/dy^2\right)$ must change sign somewhere in $y \in [0, L_y]$.

3. *Adjointness and symmetry*

It is well known that a non-singular symmetric matrix has a complete set of orthonormal eigenvectors. The eigenvectors of a *non*-symmetric matrix are complete but not orthonormal. Similarly, a self-adjoint linear differential operator has a complete set of orthonormal eigenfunctions, but the eigenfunctions of a *non*-self-adjoint operator are not orthonormal. Notions of orthonormality bias much of our perception of dynamics – for example, as regards normal modes – but meteorology teems with problems involving linear differential operators that are not self-adjoint; Held's case is just one example. Given any functions F and G which satisfy specified boundary conditions, a self-adjoint operator L in one space dimension (y) obeys

$$\int FLGdy = \int GLFdy$$

The integral is taken between specified limits, at which the boundary conditions are applied. [We have assumed the simplest "norm", and that the functions F and G are real.] Consider two examples, in each of which we require $F = G = 0$ on $y = 0, L$.

a. $\mathrm{L} \equiv \dfrac{d^2}{dy^2}$: $$F\frac{d^2G}{dy^2} = \frac{d}{dy}\left(F\frac{dG}{dy} - G\frac{dF}{dy}\right) + G\frac{d^2F}{dy^2}$$

Thus $$\int_0^L F\frac{d^2G}{dy^2}dy = \int_0^L G\frac{d^2F}{dy^2}dy \quad \text{and} \quad \mathrm{L} = d^2/dy^2 \text{ is self-adjoint.}$$

b. $\mathrm{L} \equiv U(y)\dfrac{d^2}{dy^2}$: $$FU\frac{d^2G}{dy^2} = \frac{d}{dy}\left(FU\frac{dG}{dy} - G\frac{d}{dy}(UF)\right) + G\frac{d^2}{dy^2}(UF)$$

Thus $$\int_0^L FU\frac{d^2G}{dy^2}dy = \int_0^L GU\left\{\frac{d^2}{dy^2} + \frac{2}{U}\frac{dU}{dy}\frac{d}{dy} + \frac{1}{U}\frac{d^2U}{dy^2}\right\}Fdy$$

$\mathrm{L} \equiv U(y)\dfrac{d^2}{dy^2}$ is not self-adjoint. The operator $\dfrac{1}{U}\dfrac{d^2}{dy^2}(U\,\cdot)$ is called the *adjoint* of L.

20. Large amplitude effects

Small amplitude waves and the linear equations describing them have a certain order and tractability: similar behaviour is sometimes found in apparently different cases (such as acoustic waves and shallow water waves), most of the mathematical machinery is well-established (if in need of lubrication) and analytical results can often be obtained. Large amplitude waves are usually associated with nonlinear governing equations. Such equations are more difficult to analyse than linear ones, and they present a far richer pageant of behaviour. We can offer only an overview.

Large amplitude effects in the shallow water equations

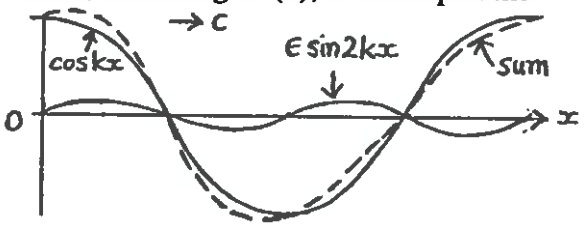
The nonlinear 1D shallow water equations can be written in the form:

$$\frac{\partial u}{\partial t} = -g\frac{\partial h'}{\partial x} - u\frac{\partial u}{\partial x} \quad ; \quad \frac{\partial h'}{\partial t} = -h_0\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}(uh') \qquad (1)$$

Here $h = h_0 + h'$, h_0 being the mean depth (taken to be independent of x). What is the effect of the nonlinear terms on a wave solution of the *linearised* versions of (1)? If $h' = \eta\cos k(x-ct)$ and $u = (g\eta/c)\cos k(x-ct)$ [with $c^2 = gh_0$] then we find that

$$-u\frac{\partial u}{\partial x} = \frac{g^2\eta^2}{2c^2}\sin 2k(x-ct) \quad ; \quad -\frac{\partial}{\partial x}(uh') = \frac{g\eta^2}{2c}\sin 2k(x-ct) \qquad (2)$$

So the effect of the nonlinear terms in (1) will be to add to u and h' terms having wavenumber $2k$ but the same phase speed as that of the linearised solution. It is readily seen that the effect of the nonlinear terms, according to (2), is to steepen the wave on its leading side, and to flatten it on its trailing other; see diagram. This argument is heuristic - we have ignored higher order interactions - but essentially correct: both shallow water and acoustic waves tend to steepen at large amplitude.



More detailed analyses show that steepening continues until *shock waves* (acoustic context) or *bores* (shallow water) are formed. The bores seen in some estuaries occur in undular and turbulent forms. Stationary bores are usually called hydraulic jumps.

General wave interactions

The effect analysed above might be called a self-interaction (of the primary cos$k(x-ct)$ wave). What if waves of different wavenumbers interact? Consider a typical nonlinear interaction without reference to a particular physical context. Suppose that

$$F = A\cos[k_1x - \omega_1t + \phi] + B\cos[k_2x - \omega_2t - \phi] \qquad (5)$$

Then $$-\partial F/\partial x = Ak_1\sin[k_1x - \omega_1t + \phi] + Bk_2\sin[k_2x - \omega_2t - \phi] \qquad (6)$$

Forming $-F\partial F/\partial x$ reveals self-interaction terms having wavenumbers $2k_1$ and $2k_2$ (as above). We also find an interaction term which can be written as

$$\frac{AB}{2}\left\{(k_1+k_2)\sin[(k_1+k_2)x - (\omega_1+\omega_2)t] + (k_2-k_1)\sin[(k_2-k_1)x - (\omega_2-\omega_1)t - 2\phi]\right\} \qquad (7)$$

The interaction generates wavenumbers and frequencies that are the sum and difference of the input values. It is most sustained when the input waves are non-dispersive, so that $\omega_1 = ck_1$, $\omega_2 = ck_2$, and (from (7)) the sum and difference waves move at the same phase speed as the two input waves.

Linearised solutions at finite amplitude

As a counterpoise to this picture of energy spreading to different space and time scales, we mention cases in which the nonlinear terms *vanish*. Consider the BVE (L17) with a uniform zonal flow U_0, but with nonlinear wave terms retained:

$$\left(\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}\right)\nabla^2\psi' + \left(\mathbf{v}'_\psi.\nabla\right)\nabla^2\psi' + \beta\frac{\partial\psi'}{\partial x} = 0 \qquad (8)$$

In L17 we omitted the nonlinear term and found Rossby wave solutions having

$$\psi' = A\sin l_n y\cos\left(k_m(x-c_{mn}t)\right) \quad \text{with} \quad c_{mn} = U_0 - \frac{\beta}{\left(k_m^2+l_n^2\right)} \qquad (9)$$

However, (9) obeys $\nabla^2\psi' = -\left(k_m^2+l_n^2\right)\psi'$, so $\left(\mathbf{v}'_\psi.\nabla\right)\nabla^2\psi' = 0$ $(\because \mathbf{v}'_\psi.\nabla\psi' = 0)$. So *(9) is a solution, at any amplitude, of the nonlinear equation (8).* Any linear combination of Rossby waves and zonal flows having the same total wavenumber $\sqrt{\left(k_m^2+l_n^2\right)}$ is also a solution of (8). Infinite families of spatially complicated solutions can be built up in this way. Similar solutions exist for internal gravity waves. The nonlinear BVE also has localised solutions having a high/low dipole structure; these have been identified with *blocking* systems of similar structure in the real atmosphere.

Stability of finite amplitude waves

The existence of finite amplitude (FA) solutions such as the Rossby waves (9) opens up a new range of stability problems. Are the FA solutions themselves unstable to further perturbations? One usually finds that instability depends on the *scale* and the *amplitude* of the FA solution; Rossby waves and surface water waves may be unstable. We should tighten up our terminology: a *growing* wave is an amplifying wave on a simpler basic flow; an *unstable* wave is unstable to further perturbations.

Solitons

Wave motion in fluids may be viewed as a contest between dispersive, nonlinear and dissipative effects. Sometimes a balance may be achieved between nonlinear effects and one or both of the other two processes, with the result that large amplitude, steady, travelling waves occur. *Solitons* are a wide class of isolated disturbances which satisfy certain nonlinear equations, to which some fluid dynamical problems reduce in certain limits. One such equation is the *Korteweg- de Vries* equation:

$$\frac{\partial u}{\partial t} + (1+u)\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \qquad (10)$$

(which represents a balance between nonlinearity and dispersion). The first scientific recognition of this type of wave motion was by J. Scott Russell in 1834: he observed a "great wave of translation" on a Scottish canal. Solitons have various remarkable properties; if two collide they re-emerge unscathed (but moved!) after an interval of confusion. Solitons, and the FA solutions noted above, show that nonlinear equations do not necessarily imply a disorderly cascade of energy between different scales.

Weakly nonlinear analyses

A formal way of studying nonlinear effects is to examine behaviour just above an instability threshold. Growth rates are then very small, and it is usually possible to separate the timescales associated with the growth of the wave and with the wave itself; this separation can form the basis for a systematic power series expansion of the governing equations, which enables lowest order nonlinear effects to be analyzed. Similar methods may be used, indeed, in other circumstances, and have been applied to derive the soliton equations noted in the previous paragraph.

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